LOWER BOUNDS OF THE SLOPE OF FIBRED THREEFOLDS

MIGUEL A. BARJA

Departament de Matemàtica Aplicada I.

ETSEIB-Facultat de Matemàtiques i Estadística.

Universitat Politècnica de Catalunya.

Diagonal 64.

08028 Barcelona. Spain.

e-mail: barja@ma1.upc.es.

1991 Mathematics Subject Classification: Primary 14J10, Secondary 14D25

Abstract

We study from a geographical point of view fibrations of threefolds over smooth curves $f : T \longrightarrow B$ such that the general fibre is of general type. We prove the non-negativity of certain relative invariants under general hypotheses and give lower bounds for $K^3_{T/B}$ depending on other relative invariants. We also study the influence of the relative irregularity $q(T) - g(B)$ on these bounds. A more detailed study of the lowest cases of the bounds is given.

0. Introduction

We consider fibrations $f : T \longrightarrow B$ from a complex projective threefold T onto a complex smooth curve B. We always consider T to be normal, with at most canonical singularities and that f is relatively minimal, i.e. the restriction of K_T to any fibre of f is nef. We are interested in the case where a general fibre F is of general type. Given any fibration $g : \widetilde{T} \longrightarrow B$ from a smooth projective threefold $\widetilde{T}$ and with fibres of general type we can always get its relatively minimal associated fibration by divisorial contractions and flips (see [15], [27]).

Our aim is to study f from a geographical point of view and thereby to relate some numerical invariants of T, B and F. First of all, note that under our assumptions K_T (and hence $K_{T/B} = K_T - f^*K_B$) is a Weil, $\mathbb{Q}$-Cartier divisor. We can consider its associated divisorial sheaves ω_T and $\omega_{T/B}$, the canonical sheaf of T and the relative ca-

Partially supported by CICYT PS93-0790 and HCM project n. ERBCHRXCT-940557

nonical sheaf of f, respectively. Then $\mathcal{E} = f_*\omega_{T/B}$ is a locally free sheaf on B of rank $p_g(F)$. We then have the well defined numerical invariants (note that the first one may be a rational number):

$$K^3_{T/B} = K^3_T - 6K^2_F(b-1)$$
$$\Delta_f = \deg\mathcal{E}$$
$$\chi_f = \chi(\mathcal{O}_F)\chi(\mathcal{O}_B) - \chi(\mathcal{O}_T).$$

From [10] we know that $\mathcal{E}$ is a nef vector bundle and hence that $\Delta_f \geq 0$. From the nefness of direct images of multiples of the relative dualizing sheaf, it follows that $K_{T/B}$ is also nef (see [29], Theorem 1.4) and so $K^3_{T/B} \geq 0$. Moreover, if $K^3_{T/B} = 0$, then f is isotrivial (cf. [29] Corollary 1.5). When $\Delta_f = 0$ we can also deduce the isotriviality of f under some additional hypotheses (see Lemma 1.2). In general it is not known whether $\chi_f \geq 0$ holds. In §1 we give examples for which χ_f is negative (see Remark 1.7) and prove (see Theorem 1.6):

Theorem 0.1 *If the Albanese dimension of T is not equal to one, then $\chi_f \geq 0$ under some extra mild hypotheses.*

Then we can define for a wide class of fibrations two different *slopes*, $\lambda_1(f) = K^3_{T/B}\big/\Delta_f$ (when $\Delta_f \neq 0$) and $\lambda_2(f) = K^3_{T/B}\big/\chi_f$ (when $\chi_f > 0$) and prove some natural invariance of them under certain operations.

With these notations we can state the only known general result on the geography of fibred threefolds over curves, due to Ohno. A simplified version of the theorem can be stated (see [29] for a complete reference):

Theorem 0.2 ([29] Main Theorem 1, Main Theorem 2)**.** *Let $f : T \longrightarrow B$ be a relatively minimal fibration of a threefold over a smooth curve of genus b. Assume that a general fibre F is of general type.*

(i) *Assume $p_g(F) \geq 3$. Then $K^3_{T/B} \geq (4 - \varepsilon(p_g(F)))(\chi(\mathcal{O}_B)\chi(\mathcal{O}_F) - \chi(\mathcal{O}_T))$, where $\varepsilon(p_g(F)) = \frac{4}{p_g(F)}$ or $\frac{8}{p_g(F)}$ depending on whether $|K_F|$ is composed with a pencil or not.*

(ii) *If $K^3_{T/B} < 4(\chi(\mathcal{O}_B)\chi(\mathcal{O}_F) - \chi(\mathcal{O}_T))$ then F falls in a list of 7 (possible) families.*

We observe that in [29] it is not proved that χ_f is non-negative. In fact, when $\chi_f < 0$, Theorem 0.2 gives no information since $K^3_{T/B} \geq 0$ holds.

In §2 we study general lower bounds for $\lambda_1(f)$ and $\lambda_2(f)$. The main result (Theorem 2.4) reads:

Theorem 0.3. *With the same hypotheses of Theorem 0.1, if $p_g(F) \geq 3$ and $\chi_f > 0$, then $\lambda_2(f) \geq (9 - \widetilde{\varepsilon}(p_g(F)))$ except if F is fibred by hyperelliptic, trigonal or tetragonal curves, or $|K_F|$ is composed, where $\widetilde{\varepsilon}(p_g(F)) \sim O(\frac{1}{p_g(F)})$.*

In fact Theorem 2.4 is much more explicit and gives extra bounds for the exceptional cases.

Section §3 is devoted to the study of the influence of the irregularity of T on the slope. In the study of fibred surfaces we have the general inequality $\lambda(f) \geq 4 - \varepsilon(F)$ and that $\lambda(f) \geq 4$ when $q(S) > g(B)$ (cf. [36]). Similarly we get for threefolds (cf. Theorem 3.3):

Theorem 0.4. *If $q(T) > g(B)$, then $\lambda_2(f) \geq 9$ except when F has an irrational pencil of hyperelliptic, trigonal or tetragonal curves.*

Theorem 3.3 also gives explicit bounds and a structure result in the exceptional cases. The key point here is to use the condition $q(T) > g(B)$ to construct new fibrations with the same slope and with fibres of higher invariants. The result follows then from a limit process in Theorem 0.3. Also we remark that, following [2], the hypothesis $q(T) > g(B)$ can be weakened to the condition that $\mathcal{E} = f_*\omega_{T/B}$ has a locally free, rank one, degree zero quotient.

Finally in §4 we study fibrations with very low slope ($\lambda_2 < 4$). These are known to exist (cf. [29] p.664); in [29], Ohno gives a classification of them in seven possible families as stated in Theorem 0.2 (see Theorem 4.1 below for a complete description). We prove (see Theorem 4.2):

Theorem 0.5. *Let $f : T \longrightarrow B$ be a relatively minimal fibration of a normal, projective threefold T with only canonical singularities onto a smooth curve B of genus b. Assume that a general fibre F is of general type with $p_g(F) \geq 3$ and that $\chi_f = \chi(\mathcal{O}_F)\chi(\mathcal{O}_B) - \chi(\mathcal{O}_T) > 0$.*

Then, if $\lambda_2(f) < 4$, we have:

(i) $q(T) = b$

(ii) $\mathcal{E} = f_*\omega_{T/B}$ *has no invertible degree zero quotient sheaf (in particular, $\mathcal{E}$ is ample provided $b \leq 1$).*

(iii) *If $p_g(F) \geq 15$, then F has a rational pencil of curves of genus 2.*

(iv) *If $p_g(F) \leq 14$, then one of the following holds:*

  (a) *F has a rational pencil of hyperelliptic curves.*

  (b) *F has a rational pencil of trigonal curves and $q(F) = 0$.*

  (c) *F is a quintic surface in $\mathbb{P}^3$.*

In fact we have a more concrete description of case (iv) (b). Case (iv) (c) is doubtful to exist (see Remark 4.3).

The author wishes very much to thank Professor Kazuhiro Konno who pointed out the problem to him during his visit to Barcelona and for his encouragement and advice. He also wants to thank his advisor, Professor Juan Carlos Naranjo, for help and continuous support.

Notations and conventions

We work over the field of complex numbers. Varieties are always assumed to be projective, reduced and irreducible. Symbols $\sim$, $\sim_{\mathbb{Q}}$ and $\equiv$ stand for linear equivalence, $\mathbb{Q}$-linear equivalence and numerical equivalence respectively.

If $\mathcal{F}$ is a coherent sheaf on a variety X we usually put $\chi(\mathcal{F})$ instead of $\chi(X, \mathcal{F})$.

1. Slopes of fibred threefolds

Definition 1.1. Let $f : T \longrightarrow B$ be a fibration of a normal, projective threefold with only canonical singularities onto a smooth curve. Let F be a general fibre of f and put $b = g(B)$. We define

$$\Delta_f = \deg f_* w_{T/B}$$
$$\chi_f = \chi(\mathcal{O}_B)\chi(\mathcal{O}_F) - \chi(\mathcal{O}_T)$$

Lemma 1.2.

(i) $\Delta_f = \chi_f + \deg R^1 f_* \omega_{T/B} \geq \chi_f$

(ii) $\Delta_f \geq 0$. *If* $\Delta_f = 0$ *and* $|K_F|$ *is birational, then* f *is isotrivial.*

(iii) *If* $\beta : \widetilde{T} \longrightarrow T$ *is a nonsingular model of* T *and* $\widetilde{f} = f \circ \beta$, *then* $\chi_f = \chi_{\widetilde{f}}$, $\Delta_f = \Delta_{\widetilde{f}}$.

Proof.

(i) Follows from [29] Lemma 2.4 and 2.5.

(ii) $\Delta_f \geq 0$ follows from the nefness of $\mathcal{E}$ ([10]). If $\Delta_f = 0$ and $|K_F|$ is birational we can apply [17] I (see also [26], 7.64).

(iii) Canonical singularities are rational (cf. [8]) and hence $R^i \beta_* \mathcal{O}_{\widetilde{T}} = 0$ for $i \geq 1$ (cf. [16], p.50). Hence $\chi(\mathcal{O}_{\widetilde{T}}) = \chi(\mathcal{O}_T)$. The same holds for general fibres F and $\widetilde{F}$ of f and $\widetilde{f}$ respectively, so $\chi_f = \chi_{\widetilde{f}}$.

By Grauert-Riemenschneider's vanishing we have $R^i\beta_*\omega_{\widetilde{T}} = 0$ for $i \geq 1$. Hence using the spectral sequence $E_2^{p,q} = R^p f_*(R^q\beta_*\omega_{\widetilde{T}}) \Rightarrow R^{p+q}\widetilde{f}_*\omega_{\widetilde{T}}$ we obtain that for every $i \geq 0$, $R^i f_*\omega_T = R^i\widetilde{f}_*\omega_{\widetilde{T}}$ holds.

□

Definition 1.3. With the above notations if f is relatively minimal and a general fibre F of f is of general type we define:

$$\lambda_1(f) = \frac{K^3_{T/B}}{\Delta_f} \quad \text{if} \quad \Delta_f \neq 0$$
$$\lambda_2(f) = \frac{K^3_{T/B}}{\chi_f} \quad \text{if} \quad \chi_f > 0$$

Remark 1.4. In the case of fibrations of surfaces over curves we actually have $\Delta_f = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_B)\chi(\mathcal{O}_F) = -\chi_f \geq 0$ (the minus sign here is a matter of the dimension of the variety) and vanishing holds in the locally trivial case. Then when f is not locally trivial we define $\lambda(f) = K^2_{S/B}\Big/\Delta_f$.

Here we have two different possibilities for the slope of f : $K^3_{T/B}\Big/\Delta_f$ or $K^3_{T/B}\Big/\chi_f$. As we will see in §2, natural methods provide lower bounds for $\lambda_1(f)$ (hence also for $\lambda_2(f)$: see Lemma 1.5 below). Note that from the *geographical* point of view the most interesting one is $\lambda_2(f)$. But for this choice we do not know whether $\chi_f \geq 0$. The aim of this section is to show that this actually happens for general fibrations.

Lemma 1.5. *Assume $\chi_f > 0$. Then*

(i) $\lambda_2(f) \geq \lambda_1(f)$

(ii) *If $\sigma : \widetilde{B} \longrightarrow B$ is an étale map and $\widetilde{f} : \widetilde{T} = T \underset{B}{\times} \widetilde{B} \longrightarrow \widetilde{B}$ is the induced fibration, then $\lambda_i(f) = \lambda_i(\widetilde{f})$, $i = 1, 2$.*

(iii) *If $\widetilde{T} \xrightarrow{\alpha} T$ is an étale map such that $\widetilde{f} = f \circ \alpha$ has connected fibres, then $\lambda_2(f) = \lambda_2(\widetilde{f})$ (but in general $\lambda_1(f) \neq \lambda_1(\widetilde{f})$).*

Proof. (i) is obvious by Lemma 1.2 (i).

(ii) If σ does not ramify over the images of singular fibres of f then $\widetilde{T} = T \underset{B}{\times} \widetilde{B}$ is again a normal, relatively minimal threefold over $\widetilde{B}$ with only canonical singularities (cf. [26], 4.10). Clearly $K^3_{\widetilde{T}/\widetilde{B}} = (\deg\sigma)K^3_{T/B}$ and $\deg R^i\widetilde{f}_*\omega_{\widetilde{T}/\widetilde{B}} = (\deg\sigma)R^i f_*\omega_{T/B}$ by flat base change. Then $n\Delta_f = \Delta_{\widetilde{f}}$, $n\chi_f = \chi_{\widetilde{f}}$ and we are done.

(iii) By [12] Ex. 18.3.9 we have $K_{\widetilde{T}} = \alpha^*(K_T)$ and that $\chi(\mathcal{O}_{\widetilde{T}}) = (\deg\alpha)\chi(\mathcal{O}_T)$.
□

The question now is whether $\chi_f \geq 0$ holds. This is not always true (see Remark 1.7). We give criteria for its non-negativity depending on the Albanese dimension of T. By Lemma 1.2 we have $\chi_f = \chi_{\widetilde{f}}$ where $\widetilde{f} = f \circ \beta$, $\beta : \widetilde{T} \longrightarrow T$ being a desingularization. Hence we can assume T is smooth.

First of all, consider $t \in B$, such that F_t is smooth, and the Albanese maps

$$\begin{tikzcd}
F_t \arrow[r, "\mathrm{alb}_{F_t}"] \arrow[d, "i_t"] & \mathrm{Alb}(F_t) \arrow[d, "(i_t)_*"'] \\
T \arrow[r, "\mathrm{alb}_T"] \arrow[d, "f"] & \mathrm{Alb}(T) \arrow[d, "f_*"'] \\
B \arrow[r, "\mathrm{alb}_B"] & \mathrm{Alb}(B)
\end{tikzcd}$$

and let $\Sigma = \mathrm{alb}_T(T)$. We set $a = \dim(alb_T(T)) = \dim \Sigma$. Let $S \longrightarrow \Sigma$ be a minimal desingularization of Σ and $\pi : \widetilde{T} \longrightarrow S$ the induced map on a birational model of T.

Note that by rigidity, Im $(i_t)_* = A$ is an abelian variety independent of t, of dimension $q(T) - b$.

Also consider the induced map $\mathrm{Pic}^0 T \xrightarrow{(i_t)^*} \mathrm{Pic}^0(F_t)$, whose image is $\widehat{A} \hookrightarrow \mathrm{Pic}^0(F_t)$. We say that f is *special* if for general $t \in B$, $\widehat{A} \hookrightarrow (h_t)^*(\mathrm{Pic}^0 C_t) \subseteq \mathrm{Pic}^0(F_t)$, for some $h_t : F_t \longrightarrow C_t$ a fibration over a curve of genus $g(C_t) \geq 2$. Otherwise we say that f is *general.*

Theorem 1.6. *Let $f : T \longrightarrow B$ be a fibration of a normal, projective threefold with only canonical singularities onto a smooth curve of genus b. Let F be a general fibre of f. Let $a = \dim alb_T(T)$.*

Then $\chi_f = \chi(\mathcal{O}_B)\chi(\mathcal{O}_F) - \chi(\mathcal{O}_T) \geq 0$ provided one of the following conditions holds:

(i) *$b \leq 1$ and $\chi(\mathcal{O}_T) \leq 0$.*

(ii) *$a = 2$, $b \geq 1$ and $h^\circ(S, \pi_* w_{\widetilde{T}/S}) \neq 0$.*

(iii) *$a = 3$, f is special and is semistable.*

(iv) *$a = 3$ and f is general.*

Remark 1.7. Part (i) of the theorem is trivial since when $b \leq 1$, $\chi_f \geq -\chi\mathcal{O}_T$. We only want to remark that condition $\chi(\mathcal{O}_T) \leq 0$ holds in most cases. Indeed, if T is smooth and K_T ample, then $\chi(\mathcal{O}_T) = \frac{1}{24}c_1c_2 < 0$ by Miyaoka-Yau inequality. Also, if T is minimal and Gorenstein, $\chi(\mathcal{O}_T) \leq 0$ holds (cf. [25]). Finally, if $a = 3$, then $\chi(\mathcal{O}_T) \leq 0$ by a consequence of generic vanishing results (see [13]). Observe that if $a = 0$, then necessarily $q(T) = b = 0$ and hence this possibility is included in (i), i.e., we need to know whether $\chi(\mathcal{O}_T) \leq 0$. Extra conditions included for the cases $a = 2, 3$

are very mild. This is clear for the case $a = 3$. In the case $a = 2$ observe that if E is a general curve on S and H is its pullback on $\widetilde{T}$, then $h^0(S, \pi_*\omega_{\widetilde{T}/S}) = h^0(E, \pi_*\omega_{H/E})$ (see proof of part (iii) of the theorem). Then by Proposition 1.8.(i) $\pi_*\omega_{H/E}$ is nef and (in general) contains an ample vector bundle, so condition $h^0(E, \pi_*\omega_{H/E}) \neq 0$ is not very restrictive.

We want to stress that in the statement of the theorem some hypotheses are needed since $\chi_f \geq 0$ *is not always true*. Indeed, we can construct counterexamples following [25] Remark 8.7. Let (C_i, τ_i) (i=1,2,3) be smooth curves with an involution. Let $D_i = C_i/\tau_i$ and $g_i = g(C_i) \geq 2$, $b_i = g(D_i)$. Consider $X = C_1 \times C_2 \times C_3$ and $\tau : X \longrightarrow X$ the involution acting on C_i as τ_i. Consider $T = X/\tau$. Then T is a threefold of general type with a finite number of (canonical) singularities, endowed with a fibration $f : T \longrightarrow D_1 =: B$ with general fibre $F \cong C_2 \times C_3$ (hence it is isotrivial). Then:

$$\begin{aligned}
\chi(\mathcal{O}_B) &= 1 - b_1 \\
\chi(\mathcal{O}_F) &= (1 - g_1)(1 - g_2) \\
h^1(T, \mathcal{O}_T) &= q(T) = b_1 + b_2 + b_3 \\
h^2(T, \mathcal{O}_T) &= b_1b_2 + b_1b_3 + b_2b_3 + (g_1 - b_1)(g_2 - b_2) + (g_1 - b_1)(g_3 - b_3) + \\
&\quad + (g_2 - b_2)(g_3 - b_3) \\
h^3(T, \mathcal{O}_T) &= b_1b_2b_3 + (g_1 - b_1)(g_2 - b_2)b_3 + (g_1 - b_1)(g_3 - b_3)b_2 + \\
&\quad + (g_2 - b_2)(g_3 - b_3)b_1.
\end{aligned}$$

If we take $b_1 = b_2 = b_3 = 0$ (C_i must be hyperelliptic then) we obtain $a = 0$ and $\chi_f < 0$. Any base change, étale over the critical points of this fibration, to a curve of positive genus produces a new fibration with $q(\widetilde{T}) = \widetilde{b} \geq 1$ (hence with $a = 1$) and $\chi_{\widetilde{f}} < 0$.

If we take $b_1 = b_2 = 1$, $b_3 = 0$ we obtain again $\chi_f < 0$ and $q(T) > b$ (so $a \geq 2$).

Finally we must say that we do not have any reasonable criteria for the nonnegativity of χ_f when $a = 1$, which corresponds to the case when $q(T) = b$, $f_* = \mathrm{Id}$ and alb_T factors through f (f is, then, an Albanese fibration). Nevertheless this is precisely the case in which we are not interested in Theorem 3.3.

Proof. (ii) From Remark 2.3 we have that $\chi_f = \chi_{\widetilde{f}}$, where $\widetilde{f} = f \circ \beta$, $\beta : \widetilde{T} \longrightarrow T$ is a desingularization. Hence we can assume T smooth. We can always assume, by the same arguments, that $\pi : T \longrightarrow S$ has branch locus contained in a normal crossings divisor.

Since $b \geq 1$ we now have a factorization of f

$$\begin{tikzcd} T \arrow[dd, "f"'] \arrow[dr, "\pi"] & \\ & S \arrow[dl, "g"] \\ B & \end{tikzcd}$$

where g need not to be a relatively minimal fibration. Let $C_t = g^{-1}(t)$ be a general fibre and $\pi_t : F_t \longrightarrow C_t$ the induced fibration. In order to simplify the notation we sometimes will use C instead of C_t. Let G be a general fibre of π_t. Note that we have $(\pi_t)_*\omega_{F_t/C_t} = \pi_*\omega_{T/S} \otimes \mathcal{O}_{C_t}$ and hence $(R^1\pi_t)_*\mathcal{O}_{F_t} = (R^1\pi_*\mathcal{O}_T) \otimes \mathcal{O}_{C_t}$ by relative duality on C_t.

Take a n-torsion element $\mathcal{L} \in \mathrm{Pic}^0(S)$ verifying that for $1 \leq i \leq n-1$ $\mathcal{L}^{\otimes i} \notin g^*(\mathrm{Pic}^0(B))$ (this is possible since S is of Albanese general type by construction) and such that $h^\circ(C_t, R^1(\pi_t)_*\mathcal{O}_{F_t} \otimes \mathcal{L}_{|C_t}) = 0$ (this is also possible since $\{\widetilde{\mathcal{L}} \in \mathrm{Pic}^0(C_t) \,|\, h^\circ(C_t, (R^1\pi_t)_*\mathcal{O}_{F_t} \otimes \widetilde{\mathcal{L}}) \neq 0\}$ is a finite set (see Proposition 1.8) and the image of $\mathrm{Pic}^0(S) \longrightarrow \mathrm{Pic}^0(C_t)$ is a subtorus of positive dimension otherwise $q(S) = b$, a contradiction).

Let $\mathcal{M} = \pi^*\mathcal{L} \in \mathrm{Pic}^0(T)$. Since π has a normal crossings ramification locus, we have that $R^1\pi_*\mathcal{O}_T$ is locally free (cf. [19] Theorem 2.6 and §3) and hence $g_*(R^1\pi_*\mathcal{O}_T \otimes \mathcal{L})$ is torsion free (hence it is locally free since B is a smooth curve). Then:

$$g_*(R^1\pi_*\mathcal{O}_T \otimes \mathcal{L}) = 0$$

since $\mathrm{rk}\; g_*(R^1\pi_*\mathcal{O}_T \otimes \mathcal{L}) = h^\circ(C_t, R^1(\pi_t)_*\mathcal{O}_{F_t} \otimes \mathcal{L}_{|C_t}) = 0$ by the choice of $\mathcal{L}$.

Using the spectral sequence $E_2^{p,q} = R^p g_*(R^q\pi_*\mathcal{F}) \Rightarrow R^{p+q}f_*\mathcal{F}$ and that $R^2\pi_*\mathcal{O}_T = 0$ (since it is locally free, being the branch locus of π contained in a normal crossings divisor (see [19])) and $R^2 g_* = 0$ by reason of fibre dimension we have

$$R^2 f_*(\mathcal{M}) = R^1 g_*(R^1\pi_*(\mathcal{M})) = R^1 g_*(R^1\pi_*\mathcal{O}_T \otimes \mathcal{L}). \tag{1}$$

We observe that $R^2 f_*(\mathcal{M})$ is locally free. Indeed $\mathcal{M} = \pi^*\mathcal{L}$; since $\mathcal{L}$ is torsion and $\mathcal{L}_{|C}^{\otimes i} \neq \mathcal{O}_C$ for $1 \leq i \leq n-1$, we can consider the induced étale base change of π:

$$\begin{tikzcd} \widehat{T} \arrow[r] \arrow[d, "\widehat{\pi}"'] & T \arrow[d, "\pi"] \\ \widehat{S} \arrow[r] \arrow[dr, "\widehat{g}"'] & S \arrow[d, "g"] \\ & B \end{tikzcd}$$

and get

$$R^j \widehat{f}_* \omega_{\widehat{T}/B} = \overset{n-1}{\underset{i=0}{\oplus}} R^j f_*(\omega_{T/B} \otimes \mathcal{M}^{\otimes i}) \tag{2}$$

$$R^j \widehat{f}_* \mathcal{O}_{\widehat{T}} = \overset{n-1}{\underset{i=0}{\oplus}} R^j f_*(\mathcal{M}^{\otimes i}).$$

Hence $R^2 f_*(\mathcal{M})$ is locally free, being a subsheaf of $R^2 \widehat{f}_* \mathcal{O}_{\widehat{T}}$ (which is locally free by relative duality and [19]).

Finally, remember that for fibrations of surfaces over curves we have (cf. [36]; for this we do not need the fibration to be relatively minimal):

$$\deg\, (\pi_t)_* \omega_{F_t/C_t} = \chi(\mathcal{O}_F) - \chi(\mathcal{O}_C)\chi(\mathcal{O}_G); \quad \deg\, g_* \omega_{S/B} = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)\chi(\mathcal{O}_B).$$

Now we can compute

$$\begin{array}{lll} \chi(\mathcal{O}_T) & = \chi(\pi_* \mathcal{O}_T) - \chi(R^1 \pi_* \mathcal{O}_T) & \\ & = \chi(\mathcal{O}_S) - \chi(R^1 \pi_* \mathcal{O}_T \otimes \mathcal{L}) & \text{since} \quad \mathcal{L} \in \mathrm{Pic}^0(S) \\ & = \chi(\mathcal{O}_S) + \chi(R^1 g_* (R^1 \pi_* \mathcal{O}_T \otimes \mathcal{L})) & \text{by Leray} \\ & = \chi(\mathcal{O}_S) + \chi(R^2 f_*(\mathcal{M})) & \text{by (1)} \\ & = \chi(\mathcal{O}_S) + \chi(f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})^*) & \text{by relative duality} \\ & = \chi(\mathcal{O}_S) - \deg(f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})) + h^0(F, \omega_F \otimes \mathcal{M}^{-1}_{|F})\chi(\mathcal{O}_B) & \text{by R.R. on } B \end{array}$$

By the choice of $\mathcal{L}$, Serre duality on B and relative duality on B and C, we obtain

$$h^0(F, \omega_F \otimes \mathcal{M}^{-1}_{|F}) = h^0(C, \pi_* \omega_{F/C} \otimes \omega_C \otimes \mathcal{L}^{-1}_{|C}) = \chi(\pi_* \omega_{F/C} \otimes \omega_C \otimes \mathcal{L}^{-1}_{|C}) \tag{3}$$

$$\begin{aligned} &= -\chi(R^1 \pi_* \mathcal{O}_F \otimes \mathcal{L}) = -\chi(R^1 \pi_* \mathcal{O}_F) = (\chi(\mathcal{O}_F) - \chi(\mathcal{O}_C)\chi(\mathcal{O}_G)) - g(G)\chi(\mathcal{O}_C) \\ &= \chi(\mathcal{O}_F) - \chi(\mathcal{O}_C) \end{aligned}$$

Hence:

$$\chi_f = \chi(\mathcal{O}_F)\chi(\mathcal{O}_B) - \chi(\mathcal{O}_T) = \deg(f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})) - \deg(g_* \omega_{S/B}) \tag{4}$$

Now we use the hypothesis: $\pi_* \omega_{T/S}$ has a section and hence we have an injection

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \pi_* \omega_{T/S}$$

which gives

$$0 \longrightarrow \omega_{S/B} \otimes \mathcal{L}^{-1} \longrightarrow \pi_* \omega_{T/S} \otimes \omega_{S/B} \otimes \mathcal{L}^{-1} = \pi_*(\omega_{T/B} \otimes \mathcal{M}^{-1})$$

and so

$$0 \longrightarrow g_*(\omega_{S/B} \otimes \mathcal{L}^{-1}) \xrightarrow{\tau} g_*(\pi_*(\omega_{T/B} \otimes \mathcal{M}^{-1})) = f_*(\omega_{T/B} \otimes \mathcal{M}^{-1}) \tag{5}$$

Note that $\deg(g_*(\omega_{S/B} \otimes \mathcal{L}^{-1})) = \deg g_*\omega_{S/B}$. Indeed, by the choice of $\mathcal{L}$ we have that $h^1(C, \omega_C \otimes \mathcal{L}^{-1}_{|C}) = 0$; on the other hand $R^1 g_*(\omega_{S/B} \otimes \mathcal{L}^{-1})$ is locally free, being a subsheaf of $R^1\widetilde{g}_*\omega_{\widetilde{S}/B}$ for the étale cover $\widetilde{S} \longrightarrow S$ induced by $\mathcal{L}$. Hence $R^1 g_*(\omega_{S/B} \otimes \mathcal{L}^{-1}) = 0$. Since $\deg R^1 g_*(\omega_{S/B}) = \deg(\mathcal{O}_S) = 0$ we obtain the desired result using that $\sum_{j=0}^{1} (-1)^j \deg R^j g_*(\omega_{S/B} \otimes \mathcal{L})$ is independent of $\mathcal{L} \in \mathrm{Pic}^0(S)$.

In order to finish the proof, it suffices to check that $f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})$ is nef. By (2) we have that $f_*(\omega_{T/B} \otimes \mathcal{M}^{-1}) = f_*(\omega_{T/B} \otimes \mathcal{M}^{\otimes(n-1)})$ is nef since it is a quotient of a nef vector bundle.

(iii) Assume that for general $t \in B$ we have a fibration $h_t : F_t \longrightarrow C_t$. Let $\overset{\circ}{B} \subseteq B$ be a non-empty open set such that $f^\circ : \overset{\circ}{T} \longrightarrow \overset{\circ}{B}$ is smooth and for every $t \in \overset{\circ}{B}$ there exists such a h_t. We can now consider the fibration of abelian varieties $\psi : \mathrm{Alb}_{\overset{\circ}{T}/\overset{\circ}{B}} \longrightarrow \overset{\circ}{B}$. For every $t \in \overset{\circ}{B}$ we have an abelian subvariety $K_t = \ker(\mathrm{Alb}F_t \longrightarrow \mathrm{Alb}C_t) \hookrightarrow \mathrm{Alb}F_t = \psi^{-1}(t)$. Then we can apply [3] Theorem 2.5 and get, after a base change, a relative abelian subvariety $K \hookrightarrow \mathrm{Alb}_{\overset{\circ}{T}/\overset{\circ}{B}}$ over $\overset{\circ}{B}$. Let $J = \mathrm{Alb}_{\overset{\circ}{T}/\overset{\circ}{B}} \big/ K$. Consider the natural map, after a base change, $\varphi : \overset{\circ}{T} \longrightarrow \mathrm{Alb}_{\overset{\circ}{T}/\overset{\circ}{B}} \longrightarrow J$ over $\overset{\circ}{B}$. For general $t \in \overset{\circ}{B}$, $\varphi_t : F_t \longrightarrow J_t$ has as its image C_t by construction. Let $\overset{\circ}{S} = \varphi(\overset{\circ}{T})$ and complete the map to get

$$\begin{tikzcd}
& \widetilde{T} \arrow[d] \arrow[dr, "\pi"] & \\
T \arrow[d, "f"'] & \overline{T} \arrow[l, "\bar{\sigma}"'] \arrow[r] \arrow[d, "\bar{f}"'] & S \arrow[dl] \\
B & \overline{B} \arrow[l, "\sigma"] &
\end{tikzcd}$$

Note that we are in the same situation for $\bar{f}$ as in (ii). We have even more since by construction the hypothesis $h^0(\pi_*\omega_{T/S}) > 0$ holds; indeed, let E be a general curve on S and let H be its pullback on T. We have that

$$\pi_*\omega_{H/E} = (\pi_*\omega_{T/S}) \otimes \mathcal{O}_E.$$

If E is ample enough, we also have

$$h^0(S, \pi_*\omega_{T/S} \otimes \mathcal{O}_S(-E)) = h^1(S, \pi_*\omega_{T/S} \otimes \mathcal{O}_S(-E)) = 0.$$

Hence $h^0(S,\pi_*\omega_{T/S}) = h^0(E,\pi_*\omega_{H/E})$. But in the case of fibred surfaces, $h^0(E,\pi_*\omega_{H/E}) \geq q(H)-g(E)$ holds according to Fujita's decomposition (see Proposition 1.8 (i)). Finally note that by [7] $q(H)-g(E) \geq q(T)-q(S) \geq 1$.

So we can apply the same argument as in (ii) and get $\chi_{\bar{f}} \geq 0$.

Then we have

$$\begin{tikzcd} \overline{T} \arrow[r, "\alpha"] \arrow[dr, "\bar{f}"'] & T \times_B \overline{B} \arrow[r] \arrow[d, "f'"] & T \arrow[d, "f"] \\ & \overline{B} \arrow[r, "\sigma"'] & B \end{tikzcd}$$

where α is induced by $\overline{f}$ and $\overline{\sigma}$. Since f is semistable we can apply base change theorem ([26], 4.10) and get

$$\overline{f}_*\omega_{\overline{T}/\overline{B}} = f'_*\omega_{T\underset{B}{\times}\overline{B}/\overline{B}} = \sigma^*(f_*\omega_{T/B})\,.$$

In fact we also have the same equality for R^1f_*: take $\mathcal{H}$ a very ample line bundle in T and let H be a general smooth member of its associated linear system. We have in a natural way

$$0 \longrightarrow f_*\omega_{T/B} \longrightarrow f_*(\omega_{T/B}\otimes\mathcal{H}) \longrightarrow f_*\omega_{H/B} \longrightarrow R^1f_*\omega_{T/B} \longrightarrow 0$$

since $R^1f_*(\omega_{T/B}\otimes\mathcal{H}) = 0$ (by Kodaira vanishing $h^1(F,\omega_F\otimes\mathcal{H}_{|F}) = 0$ and $R^1f_*(\omega_{T/B}\otimes\mathcal{H})$ is locally free by the trick of a cyclic cover used in (ii)).

Note that all of them are locally free. Hence we have that after taking σ^* we still have a long exact sequence. Considering the analogous exact sequence for $\overline{f}$ and the natural maps we get

$$\begin{tikzcd} 0 \arrow[r] & \sigma^*(f_*\omega_{T/B}) \arrow[r] & \sigma^*(f_*\omega_{T/B}\otimes\mathcal{H}) \arrow[r] & \sigma^*(f_*\omega_{H/B}) \arrow[r] & \sigma^*(R^1f_*\omega_{T/B}) \arrow[r] & 0 \\ 0 \arrow[r] & \bar{f}_*(\omega_{\overline{T}/\overline{B}}) \arrow[r] \arrow[u, "\cong"] & \bar{f}_*(\omega_{\overline{T}/\overline{B}}\otimes\overline{\mathcal{H}}) \arrow[r] \arrow[u] & \bar{f}_*\omega_{\overline{H}/\overline{B}} \arrow[r] \arrow[u, "\cong"] & R^1\bar{f}_*\omega_{\overline{T}/\overline{B}} \arrow[r] \arrow[u, "\gamma"] & 0 \end{tikzcd}$$

where γ is naturally induced and exhaustive. Since $R^1\bar{f}_*\omega_{\overline{T}/\overline{B}}$ and $\sigma^*(R^1f_*\omega_{T/B})$ are both locally free sheaves of the same rank over $\overline{B}$, γ is an isomorphism.

So we have

$$0 \leq \chi_{\bar{f}} = \deg\bar{f}_*\omega_{\overline{T}/\overline{B}} - \deg\, R^1\bar{f}_*\omega_{\overline{T}/\overline{B}} = n(\deg f_*\omega_{T/B} - \deg R^1f_*\omega_{T/B}) = n\chi_f\,.$$

(iv) Since T is of Albanese general type, then so is F_t for $t \in B$ general. We can apply then [6] Theorem 1 to get that $\{\mathcal{L} \in \mathrm{Pic}^0(F_t) |\, h^1(F_t,\mathcal{L}) \neq 0\}$ is the union of subtori $h_i^*(\mathrm{Pic}^0(C_i))$ for fibrations $h_i : F_t \longrightarrow C_i$ with $g(C_i) \geq 2$ and a finite number of (torsion) points.

Under our assumptions we can take an n-torsion element $\mathcal{L} \in \mathrm{Pic}^0(T)$ such that for $1 \leq i \leq n-1$, $h^1(F_t,\mathcal{L}^{\otimes i}_{|F_t}) = 0$.

Hence as in Lemma 1.5 (iii) if we consider the étale cover $\sigma : \widetilde{T} \longrightarrow T$ associated to $\mathcal{L}$, and $\widetilde{f} = f \circ \sigma$ we have that $f_*(\omega_{T/B} \otimes \mathcal{L})$ is a quotient of $\widetilde{f}_*\omega_{\widetilde{T}/B}$ hence it is nef.

Since $h^1(F_t, \mathcal{L}^{-1}_{|F_t}) = 0$ we have $R^1f_*(\omega_{T/B} \otimes \mathcal{L}) = 0$ (as above it is locally free) and hence

$$\begin{aligned}\chi_f &= \deg f_*\omega_{T/B} - \deg R^1f_*\omega_{T/B} = \\ &= \deg f_*(\omega_{T/B} \otimes \mathcal{L}) - \deg R^1f_*(\omega_{T/B} \otimes \mathcal{L}) = \deg f_*(\omega_{T/B} \otimes \mathcal{L}) \geq 0\end{aligned}$$

□

We finish with the following result on the structure of $\mathcal{E} = f_*w_{T/B}$. The first part is a well known result of Fujita ([10], [11]). The second part can be found in [2]. We include a brief idea of proof for benefit of the reader.

Proposition 1.8. *Let X, Y be smooth varieties of dimensions $n > m$ respectively. Let $f : X \longrightarrow Y$ be a fibration with a simple normal crossings branching.*

Then:

(i) *If $m = 1$, $\mathcal{E} = f_*w_{X/Y} = \mathcal{A} \oplus \bigoplus_{i=1}^{r} E_i \oplus \mathcal{O}_Y^h$ where $\mathcal{A}$ is an ample vector bundle (or zero), E_i are stable, degree zero non-trivial vector bundles and $h = h^1(Y, f_*\omega_X)$. If X is a surface ($n = 2$), then $h = q(X) - q(Y)$. If E is a stable degree zero vector bundle which is a quotient of $\mathcal{E}$, then E is one of the E_i or $\mathcal{O}_Y$.*

(ii) *If there exists a vector bundle E with $\det(E) = Ł \in \mathrm{Pic}^\circ(Y)$ and an epimorphism $\mathcal{E} = f_*w_{X/Y} \longrightarrow E$, then Ł is torsion. In particular, when $m = 1$, E_i is a torsion line bundle whenever rank $(E_i) = 1$.*

Proof. (i) When Y is a curve we have a decomposition $f_*\omega_{X/Y} = \mathcal{A} \oplus \mathcal{U}$, where $\mathcal{A}$ is ample (or zero) and $\mathcal{U}$ is flat (see [11]). Since Y is a curve and $\mathcal{U}$ is flat, we have a decomposition of $\mathcal{U}$ in direct sum of stable, degree zero pieces (see, for example, [6]). Finally we can use [10].

(ii) By induction on the dimension of Y we can assume that Y is a curve. It its easy to see that Ł is a quotient of $f_*^{(s)}\omega_{X^{(s)}/Y}$ (the s-th fibred product of f over Y (see [34])). Then we can check that Ł increases the general value of $h^0(Y, (R^df_*^{(s)}\mathcal{O}^{(s)}) \otimes \mathcal{M})$, for $\mathcal{M} \in \mathrm{Pic}^0Y$. Then we can apply a result of Simpson (see [32]) to get that Ł is torsion. □

2. Lower bounds for the slopes of fibred threefolds

We give here a lower bound for $\lambda_1(f)$ (and hence for $\lambda_2(f)$ provided it is well defined) in the case of a relatively minimal fibred threefold with fibres of general type. The bounds we obtain are considerably better than Ohno's bounds ([29] Main Theorem 1) as long as $p_g(F) \gg 0$.

First we need some results on linear systems on surfaces of general type.

Lemma 2.1. *Let F be a minimal surface of general type such that $p_g(F) \geq 3$ and let $\tau : \widetilde{F} \longrightarrow F$ be a birational morphism. Let $0 \leq P \leq Q \leq \tau^* K_F$ be two nef and effective divisors, such that the complete linear systems $|P|$ and $|Q|$ are base point free. Let $r \leq s$ be the dimensions of $H^0(F, \mathcal{O}_F(P))$ and $H^0(F, \mathcal{O}_F(Q))$ respectively. Let Σ be the image of $\widetilde{F}$ through the map φ induced by $|P|$. Then:*

(i) *If φ is a generically finite map, then we have*

- $P(\tau^* K_F) \geq P^2 \geq 2r - 4 + 2q(\Sigma)$ *if φ is a double cover of a geometrically ruled surface Σ.*
- $P(\tau^* K_F) \geq P^2 \geq 3r - 7$ *otherwise.*

(ii) *If $|P|$ is composed with a pencil of curves D of (geometric) genus g, $\widehat{D} = \tau_* D$, and $|Q|$ induces a generically finite map, then we have*

- $QP \geq 2(r-1)$.
- $QP \geq 3(r-1)$ *except if D is hyperelliptic and $|Q|_{|D} = g_2^1$.*
- $QP \geq 4(r-1)$ *except if D is hyperelliptic or trigonal and $|Q|_{|D} = g_2^1$ or g_3^1.*
- $QP \geq 5(r-1)$ *except if D is hyperelliptic, trigonal or tetragonal and $|Q|_{|D} = g_2^1$, $2g_2^1$, g_3^1 or g_4^1, or D is of genus 2 or 3.*
- $P(\tau^* K_F) \geq (2g-2)(r-1)$ *or $(2g-2)r$, according to whether the pencil is rational or not, if the pencil $|\widehat{D}|$ in F has no base point.*
- $P(\tau^* K_F) \geq (2p_a(\widehat{D}) - 2 - \widehat{D}^2)(r-1)$ *if the pencil $|\widehat{D}|$ has some base point.*

(iii) *If $|K_F|$ is composed with a pencil which general member D is as in (ii), then we have*

- $P(\tau^* K_F) \geq (2g-2)(r-1)$ *or $(2g-2)r$, according to whether the pencil is rational or not, if the pencil $|\widehat{D}|$ in F has no base point.*
- $P(\tau^* K_F) \geq \max\{\sqrt{2(g-1)\left(1 - \frac{1}{p_g(F)}\right)(p_g(F)-1)\widehat{D}^2},$ $(2p_a(\widehat{D}) - 2 - \widehat{D}^2)(r-1)\}$ *otherwise.*

Proof. (i) Since P is nef and $P \leq \tau^* K_F$, we obviously have $P(\tau^* K_F) \geq P^2$. It is a well known fact that deg $\Sigma \geq r - 2 + q(\Sigma)$ if Σ is geometrically ruled and that deg $\Sigma \geq 2r - 4$ otherwise (see [4]).

Let $a =$ deg φ. If $a \geq 3$ then $P^2 \geq 3\mathrm{deg}\Sigma \geq 3(r-2) > 3r - 7$. If $a = 2$ and Σ is not geometrically ruled, then $P^2 \geq 2(2r-4) = 4r - 8 \geq 3r - 7$. If Σ is geometrically ruled, then $P^2 \geq 2\mathrm{deg}\Sigma \geq 2r - 4 + 2q(\Sigma)$.

If $a = 1$, let $C \in |P|$ be a smooth curve ($|P|$ has no base point). Then $2P_{|C} \leq (\tau^* K_F + P)_{|C} \leq (K_{\widetilde{F}} + P)_{|C} = K_C$. So deg$P_{|C} \leq g(C) - 1$. We can then apply "Clifford

plus" lemma (cf. [5]) and get $P^2 = \deg P_{|C} \geq 3h^0(C, P_{|C}) - 4 \geq 3h^0(\widetilde{F}, P) - 7 = 3r - 7$.

(ii) Let $\varphi_P(F) = C \subseteq \mathbb{P}^{r-1}$. The map $F \longrightarrow C$ may not have connected fibres; consider the Stein factorization of φ_P, $F \longrightarrow \widetilde{C} \longrightarrow C$. Note that then we have $P \equiv \alpha D$ where D is an irreducible smooth curve such that $D^2 = 0$ and $\alpha = \alpha_1\alpha_2$ where $\alpha_1 = \deg(\widetilde{C} \longrightarrow C)$ and $\alpha_2 \geq r-1$ (and equality holds only when C is rational). The pencil $|P|$ is said to be rational if $\widetilde{C} = \mathbb{P}^1$ and irrational otherwise. Note that in general also $\alpha \geq r-1$ and $\alpha \geq r$ if the pencil is irrational.

Since $P \leq Q$, the map φ_P factors through φ_Q. Let $\Sigma = \varphi_Q(F)$ and consider the induced map $\psi : \Sigma \longrightarrow C$. By construction clearly $\varphi_Q(D) \subseteq \psi^{-1}(t)$ for $t \in C$ (note that $\psi^{-1}(t)$ does not need to be connected).

We have that $QP = \alpha_2(\alpha_1 QD) \geq (r-1)(\alpha_1 QD)$. Let a be the degree of $\varphi_{Q|D}$, $\overline{D} = \varphi_Q(D)$ and $d = \deg\overline{D}$. Note that a divides $\deg\varphi_Q$ although we will not use it. We have then $\alpha_1 QD = \alpha_1 ad$. Note that $ad \geq 2$ (otherwise $\widetilde{F}$ would be covered by rational curves) and hence $QP \geq 2(r-1)$. But if $QP < 3(r-1)$, then $\alpha_1 = 1$, $a = 2$, $d = 1$ (if $a = 1$, $d = 2$ again $\widetilde{F}$ is covered by rational curves). Hence D is hyperelliptic and $|Q|_{|D} = g^1_2$.

If $QP < 4(r-1)$, then $\alpha_1 ad \leq 3$. If $\alpha_1 ad = 2$, the previous argument holds. If $\alpha_1 ad = 3$, then $\alpha_1 = 1$, $a = 3$, $d = 1$ (if $a = 1$, $d = 3$, $\widetilde{F}$ would be covered by elliptic or rational curves, a contradiction since F is of general type). Then D is trigonal and $|Q|_{|D} = g^1_3$.

If $QP < 5(r-1)$, then $\alpha_1 ad \leq 4$ and we must only study the case $\alpha_1 ad = 4$. Four possibilities may occur. Either $\alpha_1 = 2$, $a = 2$, $d = 1$ (then D is hyperelliptic and $|Q|_{|D} = g^1_2$) or $\alpha_1 = 1$, $a = 4$, $d = 1$ (then D is tetragonal and $|Q|_D = g^1_4$) or $\alpha_1 = 1$, $a = 2$, $d = 2$ (then D is again hyperelliptic and $|Q|_D = 2g^1_2$) or $\alpha_1 = a = 1, d = 4$ (then D has at most genus 3; in particular D is also hyperelliptic or trigonal).

Finally note that $D\tau^*K_F = 2g-2$ if $|\widehat{D}|$ has no base point and $D\tau^*K_F = \widehat{D}K_F = 2p_a(\widehat{D}) - 2 - \widehat{D}^2$ (by adjunction formula on F) otherwise. Hence the result follows from $P(\tau^*K_F) = \alpha D(\tau^*K_F)$ and the previous bound of α.

(iii) The first result is analogous to (ii) keeping in mind that if $P \leq \tau^*K_F$, then $|P|$ is composed with a pencil of the same genus as $|K_F|$ and with the same general fibre.

Part of the second statement follows as in (ii). For the rest recall that from Hodge Index theorem $(K_F\widehat{D})^2 \geq K_F^2\widehat{D}^2$ and that when $|K_F|$ is composed with a pencil of genus zero, then $K_F^2 \geq 2(g-1)\left(1 - \frac{1}{p_g(F)}\right)(p_g(F) - 1)$ ([22], Lemma 3.3). □

Definition 2.2. We say that a linear pencil $|Q|$ on F is of type (r, g, p), $r \geq 2$, $g \geq 2$, $p = 0, 1$ if $|Q|$ is a complete linear system of r-gonal (but not s-gonal for $s < r$) curves of (geometric) genus g, rational if $p = 0$, irrational if $p = 1$. If $|Q|$ is a rational pencil, we call $\widehat{D}$ a generic member and $\delta = K_F\widehat{D} = 2p_a(\widehat{D}) - 2 - \widehat{D}^2$. If $|Q|$ is base

point free, then clearly $\delta = 2g-2$.

Remark 2.3. We recall the following result, originally due to Xiao ([36]) for the case of surfaces and to Ohno ([29]) for 3-folds and given in full generality by Konno in [23]. See this last paper for details.

Given $f : T \longrightarrow B$ as before, consider $\mathcal{E} = f_* w_{T/B}$ and the Harder-Narasimhan filtration of $\mathcal{E}$:

$$0 = \mathcal{E}_0 \subseteq \ldots \subseteq \mathcal{E}_{\ell-1} \subseteq \mathcal{E}_\ell = \mathcal{E}$$

and $\mu_1 > \mu_2 > \ldots > \mu_\ell \geq 0$, where $\mu_i = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$. If $r_i = \mathrm{rk}\mathcal{E}_i$, then $\deg \mathcal{E} = \sum_{i=1}^{\ell} r_i(\mu_i - \mu_{i-1})$.

Consider the induced rational maps $\varphi_i : T - -- > \mathbb{P}_B(\mathcal{E}_i)$ and let $\widetilde{T} \underset{\alpha}{\longrightarrow} T$ be a desingularization of T which eliminates the indeterminacy of the φ_i. Let $\tau = \alpha_{|\widetilde{F}} : \widetilde{F} \longrightarrow F$ the restriction of α to a general fibre $\widetilde{F}$ of $f \circ \alpha$. Then we have a sequence of nef effective Cartier divisors on $\widetilde{F}$, $P_1 < P_2 < \ldots < P_\ell \leq P_{\ell+1} \leq \tau^* K_F$ (where $P_{\ell+1}$ can be chosen to be P_ℓ or $\tau^* K_F$) such that the (projective) dimension of the linear systems $|P_i|$ are $r_i - 1$ ($r_i = \mathrm{rk}(\mathcal{E}_i)$) and such that $(P_i - \mu_i \widetilde{F})$ is nef and for any $1 \leq i < \ldots < i_n \leq i_{n+1} = \ell + 1$ and any $1 \leq m \leq n$, we get

$$K^3_{T/B} = (\mu^* K_{T/B})^3 \geq \sum_{p=1}^{m-1} (P_{i_p} + P_{i_{p+1}}) P_{i_m} (\mu_{i_p} - \mu_{i_{p+1}}) + \tag{6}$$

$$+ \sum_{p=m}^{n} (P^2_{i_p} + P_{i_p} P_{i_{p+1}} + P^2_{i_{p+1}})(\mu_{i_p} - \mu_{i_{p+1}})$$

$$K^3_{T/B} \geq \sum_{p=1}^{n} (P_{i_p} + P_{i_{p+1}})(\tau^* K_F)(\mu_{i_p} - \mu_{i_{p+1}}) \tag{7}$$

where $\mu_{\ell+1} = 0$.

In particular, for the indices $\{1, \ell\}$, we have

$$K^3_{T/B} \geq P^2_\ell \mu_1 + (P^2_\ell + P^2_{\ell+1})\mu_\ell \geq P^2_\ell(\mu_1 + 2\mu_\ell). \tag{8}$$

Now we can state the main result of this section:

Theorem 2.4. *Let T be a normal, projective threefold with only canonical singularities and let $f : T \longrightarrow B$ be a relatively minimal fibration onto a smooth curve of genus b. Let F be a general fibre.*

Assume F is of general type, $p_g(F) \geq 3$ and that $\chi_f = \chi(\mathcal{O}_F)\chi(\mathcal{O}_B) - \chi(\mathcal{O}_T) > 0$. Then:

(i) *If $|K_F|$ is not composed with a pencil and $|K_F|$ has no subpencil $|P|$ of type (r, g, p), $r = 2, 3, 4$, then*

$$\lambda_2(f) \geq \lambda_1(f) \geq 9\left(1 - \frac{17}{3p_g(F) + 10}\right)$$

(ii) *If $|K_F|$ is composed with a pencil with generic fibre $\widehat{D}$ of (geometric) genus g, then*

$$\begin{array}{ll} \lambda_2(f) \geq \lambda_1(f) \geq 4g-4 & \textit{if the pencil is irrational} \\ \lambda_2(f) \geq \lambda_1(f) \geq \frac{p_g(F)}{p_g(F)+1}(4p_a(\widehat{D})-4-2\widehat{D}^2) & \textit{if the pencil is rational} \end{array}$$

(iii) *If $|K_F|$ is not composed, has no hyperelliptic subpencils, and has subpencils $\{|Q_i|\}_{i\in I}$ of type (r_i, g_i, p_i) with $r_i = 3$ or 4 then*

$$\lambda_2(f) \geq \lambda_1(f) \geq \min_{i\in I}\{\lambda_{r_i}^{p_i}(Q_i)\},$$

where

$$\lambda_3^1(Q) = 9 - \frac{9}{4g-7} - \varepsilon_3^1(p_g(F), g)$$

$$\lambda_3^0(Q) = \begin{cases} 9 - \frac{9}{\delta-3} - \varepsilon_3^0(p_g(F), \delta) & \textit{if } \delta \geq 7, \\ 6\left(1 - \frac{17}{3p_g(F)+10}\right) & \textit{otherwise} \end{cases}$$

$$\lambda_4^1(Q) = 9 - \frac{3}{4g-9} - \varepsilon_4^1(p_g(F), g)$$

$$\lambda_4^0(Q) = \begin{cases} 9 - \frac{3}{\delta-5} - \varepsilon_4^0(p_g(F), \delta) & \textit{if } \delta \geq 7, \\ 8\left(1 - \frac{17}{3p_g(F)+10}\right) & \textit{otherwise} \end{cases}$$

(iv) *If $|K_F|$ has hyperelliptic subpencils $\{|Q_i|\}_{i\in I}$ of type $(2, g_i, p_i)$ then*

$$\lambda_2(f) \geq \lambda_1(f) \geq \min_{i\in I}\{\lambda_{r_i}^{p_i}(Q_i)\}$$

where

$$\lambda_2^1(Q) = \begin{cases} 6 - \frac{2}{2g-3} - \varepsilon_2^1(p_g(F), g) & \textit{if } g \geq 3, \\ 4\left(1 - \frac{1}{p_g(F)-1}\right) & \textit{if } g = 2 \end{cases}$$

$$\lambda_2^0(Q) = \begin{cases} 6 - \frac{4}{\delta-2} - \varepsilon_2^0(p_g(F), \delta) & \textit{if } \delta \geq 4, \\ 4\left(1 - \frac{9}{2p_g(F)+5}\right) & \textit{otherwise} \end{cases}$$

and where $\varepsilon_r^p \sim O(\frac{1}{p_g(F)})$ are the following positive functions

$$\varepsilon_4^1 = \frac{(68g-159)(36g-84)}{(4g-9)^2(3p_g(F)-7)+(68g-159)(4g-9)} \qquad \varepsilon_4^0 = \frac{(17\delta-91)(9\delta-48)}{(\delta-5)^2(3p_g(F)-7)+(\delta-5)(17\delta-91)}$$

$$\varepsilon_3^1 = \frac{36(g-2)(68g-137)}{(4g-7)^2(3p_g(F)-7)+(68g-137)(4g-7)} \qquad \varepsilon_3^0 = \frac{(9\delta-36)(17\delta-69)}{(\delta-3)^2(3p_g(F)-7)+(\delta-3)(17\delta-69)}$$

$$\varepsilon_2^1 = \frac{4(3g-5)(9g-17)}{(2g-3)^2(p_g(F)-2)+(9g-17)(2g-3)} \qquad \varepsilon_2^0 = \frac{(9\delta-24)(6\delta-16)}{(\delta-2)^2(2p_g(F)-4)+(\delta-2)(9\delta-24)}$$

Remark 2.5. The statement of Theorem 2.4 looks considerably simpler if we look at its asymtotic behaviour as $p_g(F)$ tends to infinity (in which case the functions

ε tend to be zero). This behaviour will play a special role in the next section. We also observe that even if $\chi_f < 0$, then the bounds in the theorem hold for $\lambda_1(f)$ as far as $\Delta_f \neq 0$.

Proof. We consider $\mathcal{E} = f_*\omega_{T/B}$ and its Harder-Narasimhan filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \ldots \subseteq \mathcal{E}_\ell = \mathcal{E}$$

with slopes $\mu_1 > \mu_2 > \ldots > \mu_\ell \geq 0$ and ranks $1 \leq r_1 < r_2 < \ldots < r_\ell = p_g(F)$. As in §1.2, each piece induces a Cartier divisor P_i on F such that the linear system $|P_i|$ has projective dimension at least $r_i - 1$. We define as usual $\mu_{\ell+1} = 0$, $P_{\ell+1} = P_\ell$. We observe that following Remark 2.3 we could define, if necessary, $P_{\ell+1} = \tau^* K_F \geq P_\ell$ (where $|P_\ell|$ is the moving part of $|K_F|$), although this possibility will only be used in very special computations and will be specifically pointed out. Remember that we have $\Delta_f = \deg\mathcal{E} = \sum_{i=1}^{\ell} r_i(\mu_i - \mu_{i+1})$.

Consider first the case where $|K_F|$ is composed. Using Lemma 2.1 (iii), Remark 2.3 and that $r_{i+1} \geq r_i + 1$ we get, if the pencil is irrational

$$\begin{aligned}
K_{T/B}^3 &\geq \sum_{i=1}^{\ell}(P_i + P_{i+1})(\tau^* K_F)(\mu_i - \mu_{i+1}) \\
&\geq \sum_{i=1}^{\ell-1}((4g-4)r_i + (2g-2))(\mu_i - \mu_{i+1}) + (4g-4)r_\ell\mu_\ell \\
&= (4g-4)\Delta_f + (2g-2)(\mu_1 - \mu_\ell) \geq (4g-4)\Delta_f.
\end{aligned}$$

If the pencil is rational and $\widehat{D}$ is a generic member of its moving part, then we have

$$\begin{aligned}
K_{T/B}^3 &\geq \sum_{i=1}^{\ell-1}((4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)r_i - (2p_a(\widehat{D}) - 2 - \widehat{D}^2))(\mu_i - \mu_{i+1}) \\
&\quad +(4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)(r_\ell - 1)\mu_\ell \\
&= (4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)\Delta_f - (2p_a(\widehat{D}) - 2 - \widehat{D}^2)(\mu_1 + \mu_\ell).
\end{aligned}$$

By Remark 2.3 using the indices $\{1, \ell\}$, we get

$$\begin{aligned}
K_{T/B}^3 &\geq (P_1 + P_\ell)(\tau^* K_F)(\mu_1 - \mu_\ell) + 2P_\ell(\tau^* K_F)\mu_\ell \\
&\geq P_\ell(\tau^* K_F)(\mu_1 + \mu_\ell) \\
&\geq (2p_a(\widehat{D}) - 2 - \widehat{D}^2)p_g(F)(\mu_1 + \mu_\ell).
\end{aligned}$$

And hence eliminating $(\mu_1 + \mu_\ell)$ from the above inequalities, we get

$$\left(1 + \frac{1}{p_g(F)}\right) K_{T/B}^3 \geq (4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)\Delta_f$$

which proves (ii).

From now on we assume that $|K_F|$ is not composed. Let

$$m = \min\{k \mid |P_k| \quad \text{induces a generically finite map}\} \leq \ell.$$

By Remark 2.3 we have

$$K^3_{T/B} \geq \sum_{i=1}^{m-1} (P_i + P_{i+1})P_m(\mu_i - \mu_{i+1}) + \sum_{i=m}^{n} (P_i^2 + P_iP_{i+1} + P_{i+1}^2)(\mu_i - \mu_{i+1}).$$

Note that, for $i \geq m$, we have $P_{i+1}^2 \geq P_iP_{i+1} \geq P_i^2$ and, if $P_iP_{i+1} = P_i^2$, then $P_i = P_{i+1}$. Indeed, we have $P_{i+1} = P_i + D_i$, with $D_i \geq 0$. Hence $P_{i+1}^2 = P_{i+1}(P_i + D_i) \geq P_{i+1}P_i = (P_i + D_i)P_i \geq P_i^2$ since P_i and P_{i+1} are nef and D_i effective. If $P_iP_{i+1} = P_i^2$, we would have $P_iD_i = 0$. Since $|P_i|$ is base point free and is not composed, Hodge Index Theorem applies and hence either $D_i^2 < 0$ (which is impossible since then $P_{i+1}^2 = P_i^2 + 2P_iD_i + D_i^2 < P_i^2$) or $D_i = 0$. So we get $P_i = P_{i+1}$.

Assume $|K_F|$ has no hyperelliptic subpencil (in particular, the maps induced by the linear systems $|P_i|$ are never double covers of geometrically ruled surfaces). Then:

$$\text{(9)} \qquad \begin{array}{lll} \text{for} & m \leq i \leq \ell - 1, & P_i^2 + P_iP_{i+1} + P_{i+1}^2 \geq 9r_i - 17 \\ \text{and for} & i = \ell, & P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 \geq 9r_\ell - 21. \end{array}$$

Indeed, we denote by φ_i the map induced by $|P_i|$ and put $a_i = \deg\varphi_i$. Note that $r_i \geq 3$. First consider the case $m \leq i \leq \ell - 1$. By Lemma 2.1, if φ_i and φ_{i+1} are not double covers of geometrically ruled surfaces, we have $P_i^2 \geq 3r_i - 7$ and $P_{i+1}^2 \geq 3r_{i+1} - 7 \geq 3r_i - 4$; if $P_i \neq P_{i+1}$, then $P_iP_{i+1} > P_i^2 \geq 3r_i - 7$ and we are done. If $P_i = P_{i+1}$, then $(P_i^2 + P_iP_{i+1} + P_{i+1}^2) = 3P_{i+1}^2 \geq 9r_{i+1} - 21 \geq 9r_i - 12$.

If $i = \ell$ the result follows immediately from the previous considerations.

Similarly, if $|K_F|$ admits hyperelliptic subpencils,

$$\text{(10)} \qquad \begin{array}{lll} \text{for} & m \leq i \leq \ell - 1, & P_i^2 + P_iP_{i+1} + P_{i+1}^2 \geq 6r_i - 9 \\ \text{and for} & i = \ell, & P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 \geq 6r_\ell - 12. \end{array}$$

Indeed, $9r_i - 17 \geq 6r_i - 9$ (since $r_i \geq 3$) so we only have to check the case $a_i = 2$ and $\varphi_i(\widetilde{F})$ a geometrically ruled surface. Since φ_i factorizes through φ_{i+1}, $a_{i+1} = 1$ or 2. In any case $P_{i+1}^2 \geq 2r_{i+1} - 4 \geq 2r_i - 2$. If $P_i \neq P_{i+1}$, we have $P_iP_{i+1} > P_i^2 \geq 2r_i - 4$ and we are done. If $P_i = P_{i+1}$, then $P_i^2 + P_{i+1}P_i + P_{i+1}^2 = 3P_{i+1}^2 \geq 6r_i - 6$. For $i = \ell$ the assertion is clear.

Observe that since $P_1 \leq \ldots \leq P_{m-1}$, all the maps φ_i induced by $|P_i|$ are composed of the same pencil (with the only exception of $r_1 = 1$, $P_1 = 0$ for which we have no defined map φ_1). Indeed if $i < j \leq m - 1$ the map φ_i factors through the map φ_j.

Since $\varphi_i(\widetilde{F})$ and $\varphi_j(\widetilde{F})$ are curves, both maps have, after the Stein factorization, the same fibre.

Let P_{m-1} (and hence P_i for $i \le m-1$) be of type (r,g,p). Now, if $i = m-1$, then:

$$(P_{m-1}+P_m)P_m \ge \begin{cases} 10r_{m-1}-10 & \text{if} \quad r \ge 5, \\ 8r_{m-1}-8 & \text{if} \quad r = 4, \\ 6r_{m-1}-6 & \text{if} \quad r = 3, \\ 4r_{m-1}-4 & \text{if} \quad r = 2. \end{cases} \tag{11}$$

For this, simply note that $(P_{m-1}+P_m)P_m \ge 2P_{m-1}P_m$ since $P_{m-1} \le P_m$ and P_m is nef. Then apply Lemma 2.1. Note that even if $r_{m-1} = r_1 = 1$ (hence $P_1 = 0$) $(P_{m-1}+P_m)P_m \ge 10r_{m-1}-10$ holds.

Finally, if $1 \le i \le m-2$, then

$$(P_i+P_{i+1})P_m \ge \begin{cases} 10r_i-5 & \text{if} \quad r \ge 5, \\ 8r_i-4 & \text{if} \quad r = 4, \\ 6r_i-3 & \text{if} \quad r = 3, \\ 4r_i-2 & \text{if} \quad r = 2, \end{cases} \tag{12}$$

which follows immediately from Lemma 5.9, even if $P_i = P_1 = 0$ $(r_1 = 1)$.

If $\Delta_f = \deg \mathcal{E} = \sum_{i=1}^{\ell} r_i(\mu_i - \mu_{i+1})$, call $\Delta_1 = \sum_{i=1}^{m-1} r_i(\mu_i - \mu_{i+1})$ and $\Delta_2 = \Delta_f - \Delta_1$.

Let us prove first (i) and (iii); we can assume then that $|K_F|$ has no hyperelliptic subpencil. We get the following inequalities using (9),(11) and (12):

If $r \ge 5$,

$$K^3_{T/B} \ge 10\Delta_1 + 9\Delta_2 - 5\mu_1 - 5\mu_{m-1} - 7\mu_m - 4\mu_\ell \ge 9\Delta_f - 17\mu_1 - 4\mu_\ell. \tag{13}$$

If $r = 4$,

$$K^3_{T/B} \ge 8\Delta_1 + 9\Delta_2 - 4\mu_1 - 4\mu_{m-1} - 9\mu_m - 4\mu_\ell \ge 8\Delta_1 + 9\Delta_2 - 17\mu_1 - 4\mu_\ell.$$

If $r = 3$,

$$K^3_{T/B} \ge 6\Delta_1 + 9\Delta_2 - 3\mu_1 - 3\mu_{m-1} - 11\mu_m - 4\mu_\ell \ge 6\Delta_1 + 9\Delta_2 - 17\mu_1 - 4\mu_\ell.$$

Note that the bound for $r \ge 5$ also holds for $m = 2$, $P_{m-1} = P_1 = 0$ $(r_1 = 1)$, or $m = 1$. Using now Remark 2.3 and Lemma 2.1 (i), we have

$$K^3_{T/B} \ge P_\ell^2(\mu_1 + 2\mu_\ell) \ge (3p_g(F) - 7)(\mu_1 + 2\mu_\ell)$$

and so (we use $-17\mu_1 - 4\mu_\ell \ge -17(\mu_1 + 2\mu_\ell)$; note that μ_ℓ may be zero):

$$\text{If} \quad r \ge 5 \quad \left(1 + \frac{17}{3p_g(F) - 7}\right) K^3_{T/B} \ge 9\Delta_f. \tag{14}$$

$$\text{If} \quad r=4 \quad \left(1+\frac{17}{3p_g(F)-7}\right) K^3_{T/B} \geq 8\Delta_1 + 9\Delta_2.$$
$$\text{If} \quad r=3 \quad \left(1+\frac{17}{3p_g(F)-7}\right) K^3_{T/B} \geq 6\Delta_1 + 9\Delta_2.$$

The first inequality proves (i) and holds also when $m=2, \quad P_{m-1}=P_1=0 \quad (r_1=1)$, or $\quad m=1$.

In order to prove (iii) we can assume from now on that $r=3$ or 4, otherwise we have (i) which is stronger than (iii).

We can also assume $m \geq 2$ and, as pointed out, that $|P_{m-1}|$ is composed with a pencil.

We divide the argument according to whether the pencil $|P_{m-1}|$ is irrational or not.

If the pencil is rational we use the same notation as in Lemma 2.1 and Definition 2.2 and set $\widehat{D}$ for the (possibly singular) general element of the linear system $|\tau_* P_i| = |Q_i|$ in F (possibly with base points).

Then using Lemma 2.1 and according to whether the pencil is irrational or not we have

$$\begin{array}{ll}
\text{for} & i \leq m-2, \\
(15) & (P_i+P_{i+1})(\tau^* K_F) \geq (4g-4)r_i + (2g-2) \quad (\text{except if } P_i=P_1=0,\ r_1=1) \\
\text{and} & (P_i+P_{i+1})(\tau^* K_F) \geq (4p_a(\widehat{D})-4-2\widehat{D}^2)r_i - (2p_a(\widehat{D})-2-\widehat{D}^2), \\
\text{for} & i=m-1, \\
 & (P_{m-1}+P_m)(\tau^* K_F) \geq 2P_{m-1}(\tau^* K_F) \geq (4g-4)r_{m-1} \\
\text{and} & (P_{m-1}+P_m)(\tau^* K_F) \geq 2P_{m-1}(\tau^* K_F) \geq (4p_a(\widehat{D})-4-2\widehat{D}^2)(r_{m-1}-1) \\
\text{for} & m \leq i \leq \ell-1, \\
 & (P_i+P_{i+1})(\tau^* K_F) \geq P_i^2 + P_{i+1}^2 \\
\text{and for} & i=\ell \\
 & (P_\ell + P_{\ell+1}^2)(\tau^* K_F) \geq 2P_\ell^2.
\end{array}$$

Using Remark 2.3 we know

$$K^3_{T/B} \geq \sum_{i=1}^{\ell} (P_i+P_{i+1})(\tau^* K_F)(\mu_i - \mu_{i+1})$$

and so we can conclude

$$\begin{array}{ll}
\text{If} & r=3,4,\ p=1 \\
(16) & K^3_{T/B} \geq (4g-4)\Delta_1 + 6\Delta_2 - 11\mu_m - 3\mu_\ell \geq (4g-4)\Delta_1 + 6\Delta_2 - 11\mu_1 - 3\mu_\ell.
\end{array}$$

Finally:

$$\begin{aligned}
&\text{if}\quad r=3,4,\ p=0\\
&K^3_{T/B}\geq (4p_a(\widehat{D})-4-2\widehat{D}^2)\Delta_1+6\Delta_2-(2p_a(\widehat{D})-2-\widehat{D}^2)(\mu_1-\mu_{m-1})\\
&-(4p_a(\widehat{D})-4-2\widehat{D}^2)(\mu_{m-1}-\mu_m)-11\mu_m-3\mu_\ell\\
&\geq (2p_a(\widehat{D})-2-\widehat{D}^2)\Delta_1+6\Delta_2-11\mu_1-3\mu_\ell
\end{aligned}\tag{17}$$

since $\Delta_1\geq(\mu_1-\mu_{m-1})+2(\mu_{m-1}-\mu_m)$ (this is immediate if $m-1\geq 2$; if $m=2$, then $r_1\geq 2$ and $\Delta_1=r_1(\mu_1-\mu_2)\geq 2(\mu_1-\mu_m)$).

Note also that these formulas include the possibility $r_1=1,\ P_1=0$.

Using now that $K^3_{T/B}\geq(3p_g(F)-7)(\mu_1+2\mu_\ell)$, we get

$$\left(1+\frac{11}{3p_g(F)-7}\right)K^3_{T/B}\geq\begin{cases}(4g-4)\Delta_1+6\Delta_2 & \text{if } r=3,4,\ p=1\\ (2p_a(\widehat{D})-2-\widehat{D}^2)\Delta_1+6\Delta_2 & \text{if } r=3,4,\ p=0.\end{cases}$$

Considering simultaneously this last inequality together with (14) and using that $\Delta_2=\Delta_f-\Delta_1$, we get (iii). All the arguments work similarly so we only give details of one of them. Assume P_{m-1} is a tetragonal irrational pencil. Then the last inequality and (14) give

$$\left(1+\frac{17}{3p_g(F)-7}\right)K^3_{T/B}\geq 8\Delta_1+9\Delta_2=8\Delta_f+\Delta_2,$$

$$\left(1+\frac{11}{3p_g(F)-7}\right)K^3_{T/B}\geq(4g-4)\Delta_1+6\Delta_2=(4g-4)\Delta_f-(4g-10)\Delta_2.$$

Observe that since $|P_{m-1}|$ is not an hyperelliptic pencil, then $g\geq 3$ and so we can get a lower bound for Δ_2 from the second inequality and we obtain

$$\left[1+\frac{17}{3p_g(F)-7}+\frac{1}{4g-10}\left(1+\frac{11}{3p_g(F)-7}\right)\right]K^3_{T/B}\geq\left[8+\frac{4g-4}{4g-10}\right]\Delta_f$$

which gives $\lambda^1_4(P_{m-1})$. As for the computation of $\lambda^0_4(P_{m-1})$ or $\lambda^0_3(P_{m-1})$ we only must be careful when $\delta\leq 6$ since then the corresponding second inequality does not give a lower bound for Δ_2. If this case occurs, then just deduce from (14)

$$\left(1+\frac{17}{3p_g(F)-7}\right)K^3_{T/B}\geq 8\Delta_f\qquad\text{if the pencil is tetragonal}$$

$$\left(1+\frac{17}{3p_g(F)-7}\right)K^3_{T/B}\geq 6\Delta_f\quad\text{if the pencil is trigonal}$$

which give the special values of λ_r^0 in (iii).

Of course we must consider all the possibilities for $|P_{m-1}|$ being trigonal or tetragonal subpencils of $|K_F|$ and so we must consider the minimum of all such lower bounds.

Finally we must prove (iv). Assume $|K_F|$ has hyperelliptic subpencils. Then it may happen that for some $i \geq m$, φ_i is of degree two onto a ruled surface. Also may happen that $r = 2$. Altogether, Remark 2.3, Lemma 2.1 and inequalities (13), (14), (16) and (17) read

$$K_{T/B}^3 \geq P_\ell^2(\mu_1 + 2\mu_\ell) \geq (2p_g(F) - 4)(\mu_1 + 2\mu_\ell).$$

If $\quad r \geq 3 \quad$ or $\quad m = 1 \quad$ or $\quad m = 2 \quad P_{m-1} = P_1 = 0$,

$$K_{T/B}^3 \geq 6\Delta_1 + 6\Delta_2 - 3\mu_1 - 3\mu_{m-1} - 3\mu_m - 3\mu_\ell \geq 6\Delta_f - 9\mu_1 - 3\mu_\ell$$

and so

$$\text{if} \quad r \geq 3 \quad \left(1 + \frac{9}{2p_g(F) - 4}\right) K_{T/B}^3 \geq 6\Delta_f. \tag{18}$$

If $\quad r = 2$,

$$K_{T/B}^3 \geq 4\Delta_1 + 6\Delta_2 - 2\mu_1 - 2\mu_{m-1} - 5\mu_m - 3\mu_\ell \geq 4\Delta_1 + 6\Delta_2 - 9\mu_1 - 3\mu_\ell.$$

And so, if $r = 2$, then

$$\left(1 + \frac{9}{2p_g(F) - 4}\right) K_{T/B}^3 \geq 4\Delta_1 + 6\Delta_2. \tag{19}$$

If $r = 2$ $p = 0$,

$$K_{T/B}^3 \geq (2p_a(\widehat{D}) - 2 - \widehat{D}^2)\Delta_1 + 4\Delta_2 - 6\mu_1 - 2\mu_\ell.$$

If $\quad r = 2$ $p = 1$,

$$K_{T/B}^3 \geq (4g - 4)\Delta_1 + 4\Delta_2 - 2\mu_m - 2\mu_\ell \geq (4g - 4)\Delta_1 + 4\Delta_2 - 2\mu_1 - 2\mu_\ell.$$

This last inequality needs an extra explanation for the coefficient of μ_m. If $r = 2$, we have an hyperelliptic pencil on $\widetilde{F}$. Let D be a general irreducible member. For $i \geq m$, $|P_i|_D$ is a base point free sublinear system of $|K_D|$ by adjunction and hence

maps D onto $\mathbb{P}^1$. Hence, $\Sigma_i = \varphi_i(\widetilde{F})$ is always a ruled surface. Moreover, φ_i is of degree at least two. Since φ_{m-1} factors through Σ_i for $i \geq m$, then $q(\Sigma_i) \geq 1$ (the pencil is irrational). Hence for $i \geq m$:

$$P_i(\tau^* K_F) \geq P_i^2 \geq (\deg\varphi_i)(r_i - 2 + q(\Sigma_i)) \geq 2r_i - 2\,.$$

From here we get

$$(20) \qquad \left(1 + \frac{2}{2p_g(F) - 4}\right) K^3_{T/B} \geq (4g-4)\Delta_1 + 4\Delta_2 \qquad \text{if } r = 2,\ p = 1$$

$$\text{and} \qquad \left(1 + \frac{6}{2p_g(F) - 4}\right) K^3_{T/B} \geq (2p_a(\widehat{D}) - 2 - \widehat{D}^2)\Delta_1 + 4\Delta_2 \qquad \text{if } r = 2,\ p = 0.$$

If $r = 2$ (i.e., $|P_{m-1}|$ is one of the hyperelliptic subpencils of $|K_F|$) then we can proceed as in (iii) using (19) and (20) and inequalities in (iv) follow. Here the exceptional bounds appear in the rational and irrational cases. When $p = 0$ and $\delta \leq 4$, (19) gives

$$\left(1 + \frac{9}{2p_g(F) - 4}\right) K^3_{T/B} \geq 4\Delta_f$$

and so

$$\lambda_1(f) \geq 4\left(1 - \frac{9}{2p_g(F) + 5}\right).$$

When $p = 1, \quad g = 2$ we have from (20)

$$\lambda_1(f) \geq 4\left(1 - \frac{1}{p_g(F) - 1}\right).$$

If $r \geq 3$ or $|P_{m-1}|$ is not composed with a pencil the situation can only be better; indeed, in this case inequality (18) holds, which is better than inequality (19) and hence it is better than any inequalities coming from (19) as those in (iv) are. □

Remark 2.6. Theorem 2.4 shows that there is some influence on the slope of the existence of certain special maps on the general fibre. This is precisely what is known to happen partially in the case of fibred surfaces, where the gonality of a general fibre seems to play a special role (cf. [33], [24], [23]).

Remark 2.7. In [1] using a different aproach the author finds some lower bounds for $\lambda_2(f)$ assuming that T is Gorenstein and either the sheaf $\mathcal{E}$ is semistable or the canonical image of F lies in few quadrics.

3. The slope of non Albanese fibred threefolds

Consider, for $t \in B$ such that F_t is smooth, the natural diagram of Albanese maps

$$\begin{array}{ccc} F_t & \xrightarrow{\mathrm{alb}_{F_t}} & \mathrm{Alb}(F_t) \\ \downarrow{\scriptstyle i_t} & & \downarrow{\scriptstyle (i_t)_*} \\ T & \xrightarrow{\mathrm{alb}_T} & \mathrm{Alb}(T) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*} \\ B & \xrightarrow{\mathrm{alb}_B} & \mathrm{Alb}(B) \end{array}$$

As t varies, the abelian subvariety $(i_t)_*\mathrm{Alb}(F_t)$ remains constant, say A, by the rigidity property of abelian varieties. Let $\alpha = \dim A$. From this we get

$$b \leq q(T) = b + \alpha \leq b + g$$

Moreover, from the diagram and the universal property of Albanese varieties it is clear to see that if $b \geq 1$, then $b = q$ if and only if $\mathrm{alb}_T(T) = B$. In this case we say that f is an *Albanese fibration*. We will say that f is a *non-Albanese fibration* if $q(T) > b$ (i.e., if $b = 0$ and $q(T) > 0$ or if $a = \dim alb_T(T) \geq 2$). We want to analyze the influence of this fact in the slope as in Xiao's result for surfaces (cf. [36] Corollary 2.1): if $q(S) > b$ then $\lambda(f) \geq 4$. Our main tool will be Theorem 2.4 together with an argument of étale covers.

First of all we need to control when étale covers of curves are d-gonal.

Lemma 3.1. *Let D be a smooth curve and $\mathcal{L} \in \mathrm{Pic}^0(D)$ a n-torsion element.*

Let $\alpha : \widetilde{D} \longrightarrow D$ the associated étale cover of degree n.

Assume $\widetilde{D}$ has a unique base point free g^1_d; then

(i) *D has a g^1_d.*

(ii) *$n|d$.*

Proof. Easy. □

Lemma 3.2. *Let F be a surface of general type and $\mathcal{L} \in \mathrm{Pic}^0(F)$ a n-torsion element. Let $\alpha : \widetilde{F} \longrightarrow F$ be the associated étale cover of degree n. Then, if n is prime and large enough*

(i) *$\widetilde{F}$ has no rational pencil of d-gonal curves ($d = 2, 3$).*

(ii) *If $\widetilde{F}$ has an irrational pencil of d-gonal curves ($d = 2, 3$) of genus g, then so has F and there exists a base change*

$$\begin{tikzcd} \widetilde{F} \arrow[r, "\alpha"] \arrow[d, "\widetilde{h}"'] & F \arrow[d, "h"] \\ \widetilde{C} \arrow[r] & C \end{tikzcd}$$

such that $\mathcal{L} = h^*(\mathcal{M}) \in h^*(\mathrm{Pic}^0(C))$ and $\widetilde{C} \longrightarrow C$ is induced by $\mathcal{M}$.

(iii) *If $\widetilde{F}$ has a pencil of tetragonal curves then, either so does F and there exists a base change diagram as in (ii) (and necessarily the pencil is irrational), or the pencil $\{\widetilde{D}\}$ is of bielliptic curves and F has a pencil $\{D\}$ of bielliptic (hence tetragonal) or hyperelliptic curves such that $\alpha^*(D) = \widetilde{D}$ (and $2g(\widetilde{D}) - 2 = n(2g(D) - 2)$) where*

$$\begin{tikzcd} \widetilde{F} \arrow[rr, "\alpha"] \arrow[dr, "\widetilde{h}"'] & & F \arrow[dl, "h"] \\ & C & \end{tikzcd}$$

Proof. Assume first $\widetilde{F}$ has a base point free pencil $\widetilde{h} : \widetilde{F} \longrightarrow \widetilde{C}$. Let $\widetilde{D}$ be a general fibre and let $D = (\alpha(\widetilde{D}))_{\mathrm{red}}$. Clearly D is smooth since $\widetilde{D}$ moves algebraically.

If $\mathcal{L}_{|D} \neq \mathcal{O}_D$, then since n is prime $\mathcal{L}_{|D}^{\otimes i} \neq \mathcal{O}_D$ for $1 \leq i \leq n-1$ and hence $\alpha^*(D)$ is a connected smooth étale cover of D containing $\widetilde{D}$ and so $\alpha^*(D) = \widetilde{D}$.

If $D^2 > 0$, by [28], $\mathcal{L}_{|D} \neq \mathcal{O}_D$ and by the previous argument $\alpha^*(D) = \widetilde{D}$, $0 = \widetilde{D}^2 = nD^2 > 0$, which is a contradiction. So necessarily we have $D^2 = 0$.

If $\mathcal{L}_{|D} = \mathcal{O}_D$ then $\alpha^*(D) = \widetilde{D}_1 + \ldots + \widetilde{D}_n$ $(\widetilde{D}_1 = \widetilde{D})$, $\widetilde{D}_i\widetilde{D}_j = 0$, $\widetilde{D}_i \neq \widetilde{D}_j$ if $i \neq j$, and hence we have a factorization

$$\begin{tikzcd} \widetilde{F} \arrow[r, "\alpha"] \arrow[d, "\widetilde{h}"'] & F \arrow[d, "h"] \\ \widetilde{C} \arrow[r, "\beta"] & C \end{tikzcd}$$

such that $\mathcal{L} \in h^*\mathrm{Pic}^0(C)$ and β is an étale cover. In particular $g(C) \geq 1$.

Now we want to explore the possibility $\widetilde{D} = \alpha^*(D)$. Since n is large enough then so is $g(\widetilde{D})$ and hence, if $\widetilde{D}$ has a g^1_d $(d = 2, 3, 4)$ it is unique except if $d = 4$ and $\widetilde{D}$ is bielliptic. So Lemma 3.1 (ii) says that $\widetilde{D}$ has no g^1_d $(d = 2, 3, 4)$ as long as n does not divide d except when $\widetilde{D}$ is bielliptic.

But in this case, let σ be the (unique, if $g(\widetilde{D}) \geq 6$) bielliptic involution of $\widetilde{D}$. Let φ be the rank n automorphism of $\widetilde{D}$ induced by α. Since σ is unique we must have that $\varphi \circ \sigma \circ \varphi^{-1} = \sigma$ and hence there exists an automorphism $\bar{\varphi}$ of the elliptic base curve E

such that the following diagram commutes

$$\begin{tikzcd} \widetilde{D} \arrow[r, "\varphi"] \arrow[d, "2:1"'] & \widetilde{D} \arrow[d, "2:1"] \\ E = \widetilde{D}_{/<\sigma>} \arrow[r, "\bar{\varphi}"] & E \end{tikzcd}$$

Hence there is an induced degree two map $D = \widetilde{D}_{/<\varphi>} \longrightarrow E_{/<\bar{\varphi}>} = E'$, where E' is $\mathbb{P}^1$ or an elliptic curve according $\bar{\varphi}$ has fixed points or not. Then D is hyperelliptic or bielliptic (hence tetragonal). $\square$

Then we can consider what is the influence of the irregularity of T in the slope of f. We have an exceptionally good behaviour:

Theorem 3.3. *Let $f : T \longrightarrow B$ be a relatively minimal fibration of a normal, projective threefold with only canonical singularities onto a smooth curve of genus b. Let F be a general fibre. Assume F is of general type, $p_g(F) \geq 3$, and that $\chi_f = \chi(\mathcal{O}_F)\chi(\mathcal{O}_B) - \chi(\mathcal{O}_T) > 0$.*

Then, if $q(T) > b$, we have

(i) $\lambda_2(f) \geq 4$.

(ii) *If F has no irrational pencil of d-gonal curves ($d = 2, 3, 4$) then $\lambda_2(f) \geq 9$.*

(iii) *If F has an irrational tetragonal pencil of genus g and no irrational pencil of trigonal curves nor pencil of hyperelliptic curves then $\lambda_2(f) \geq 9 - \frac{3}{4g-9}$.*

(iv) *If f has an irrational tetragonal pencil of genus g_1, an irrational trigonal pencil of genus g_2 and no hyperelliptic pencil, then*

$$\lambda_2(f) \geq \min\left\{9 - \frac{3}{4g_1 - 9}, 9 - \frac{9}{4g_2 - 7}\right\}.$$

(v) *If F has an irrational hyperelliptic pencil of genus g, then $\lambda_2(f) \geq 6 - \frac{2}{2g-3}$.*

(vi) *If F has a rational hyperelliptic pencil and none irrational, then $\lambda_2(f) \geq 6$.*

(vii) *If $\lambda_2(f) < 9$ then either F has a rational pencil of hyperelliptic curves or there exists, perhaps up to base change, a factorization of f*

$$\begin{tikzcd} T \arrow[r, dashed, "h"] \arrow[d, "f"'] & S \arrow[dl, "g"] \\ B & \end{tikzcd}$$

where S is a smooth surface fibred over B by curves C_t of genus $g(C_t) \geq 1$ and h is everywhere defined at the general fibre F_t of f, such that

- *for* $t \in B$ *general the image of* $((i_t)^* : \mathrm{Pic}^0(T) \longrightarrow \mathrm{Pic}^0(F_t))$ *lies in* $h_t^*(\mathrm{Pic}^0(C_t)) \subseteq \mathrm{Pic}^0(F_t)$.
- *for* $s \in S$ *general* $D_s = h^{-1}(s)$ *is hyperelliptic, trigonal or tetragonal (necessarily hyperelliptic or of genus* 3 *if* $\lambda_2(f) < 8$*).*

Proof. Note that (i) follows from (ii), (iii), (iv), (v) and (vi). Since $q(T) > b$ then for every $n \gg 0$ there exists a n-torsion element $\mathcal{L} \in \mathrm{Pic}^0 T \setminus f^*(\mathrm{Pic}^0(B))$ such that for $1 \leq i \leq n-1$, $\mathcal{L}_{|F}^{\otimes i} \neq \mathcal{O}_F$. Then we can construct the associated étale cover $\alpha : \widetilde{T} \longrightarrow T$ as in Lemma 1.5 (iii) and get $\widetilde{f} = f \circ \alpha$ such that $\lambda_2(f) = \lambda_2(\widetilde{f})$.

If $\widetilde{F}$ is the fibre of $\widetilde{f}$, we have an étale cover $\alpha_1 : \widetilde{F} \longrightarrow F$ and hence $p_g(\widetilde{F}) \geq \chi(\mathcal{O}_{\widetilde{F}}) = n\chi(\mathcal{O}_F) \geq n$ (note that $q(\widetilde{F}) \geq q(F) \geq 1$ since $q(T) > b$). Since we can do this process for n as large as is needed, we can take in the bounds of Theorem 2.4 limit when $p_g(F)$ goes to infinity.

Assume that $|K_F|$ is composed. We have $p_g(\widetilde{F}) \geq n$ and either the genus of the fibre of the pencil or the genus of the base curve increases, except if $q(F) = 1$ and the pencil is elliptic. When $|K_F|$ is composed the pencil can only be rational or elliptic and the genus of the fibre is at most 5 provided $p_g \gg 0$ ([5], [35]). So if n is large enough and the pencil is rational, $|K_{\widetilde{F}}|$ can not be composed. Since $\lambda_2(f) = \lambda_2(\widetilde{f})$ we can assume $|K_F|$ is not composed with a rational pencil.

Finally, if the pencil is elliptic and $q(F) = 1$, note that we can apply Theorem 2.4 (ii) and get that $\lambda_2(f) \geq 12$ if $g \geq 4$, $\lambda_2(f) \geq 8$ if $g = 3$, $\lambda_2(f) \geq 4$ if $g = 2$. So (ii), (iii), (iv), (v) and (vi) hold. From now on we assume $|K_F|$ is not composed.

When F has a fibration $h : F \longrightarrow C$ we have an induced map $\widetilde{h} = h \circ \alpha_1 : F \longrightarrow C$. This map may not have connected fibres and hence factorizes through an étale cover $\widetilde{C} \longrightarrow C$. We have two possibilities.

There may exist an unbounded sequence $\{n_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that for every i $\widetilde{h}_{n_i}$ has connected fibres over C (hence it is a fibration) or for every $n \geq n_0$, $\widetilde{h}_n$ factorizes through a non trivial étale cover $\widetilde{C}_n \longrightarrow C$.

In any case we have that, if g_n is the genus of the fibration $\widetilde{F}_n \longrightarrow \widetilde{C}_n$, $g \leq g_n \leq n(g-1)+1$, the border cases being the two extreme possibilities.

If $C = \mathbb{P}^1$ (rational pencil) then $\widetilde{C}_n = C$ for all n and the sequence $\{\delta_n\}$ is unbounded. If F has a pencil of tetragonal curves which are bielliptic, then by Lemma 3.2 (iii) again it may happen that $\{g_n\}$ is unbounded. Otherwise by using Lemma 3.2, we have that $g_n = g$ holds for all n.

If the sequence $\{g_n\}$ is bounded, since $\lim_{n\to\infty} p_g(\widetilde{F}_n) = \infty$, we can consider Theorem 2.4 (iii), (iv) and get the bounds of (iii), (iv), (v) and (vi) (note that $g_n = g$ is the worst case).

Finally assume $\{g_n\}$ (or δ_n) is unbounded. We must take limit in the bounds of Theorem 2.4 when g (or δ) and $p_g(F)$ simultaneously (and linearly) grow. In all the

cases, the limit is 9.

If F has no irrational d-gonal pencil ($d = 2, 3, 4$), neither has $\widetilde{F}$ by Lemma 5.13. If F has a rational $d-$gonal pencil ($d = 2, 3, 4$), we know yet that $\lambda_2(f) \geq 9$. So we can assume that if F verifies the hypotheses of Theorem 5.11 (i), then so does $\widetilde{F}$, and so we get $\lambda_2(f) \geq 9$ in the limit process. This proves (ii).

In order to prove (vii) note that in the previous arguments we always have $\lambda_2(f) \geq 9$ except when there exists $h_t : F_t \longrightarrow C_t$ with hyperelliptic, trigonal or tetragonal fibres (such that $g(C_t) \geq 1$ when non-hyperelliptic) and for every $\mathcal{L} \in \mathrm{Pic}^0(T)$ the étale cover $\widetilde{F}_t \longrightarrow F_t$ given by $\mathcal{L}_{|F_t}$ factorizes through an étale cover of C_t. This says that Im $((i_t)^*\mathrm{Pic}^0 T \longrightarrow \mathrm{Pic}^0(F_t))$ lies in the subtorus $h_t^*\mathrm{Pic}^0 C_t$.

In order to glue all the maps h_t we can proceed as in Theorem 1.6 (iii). □

Corollary 3.4. *With the same notations as in Theorem* 3.3*, if* $q(T) > b$ *then*

(i) *If* $\lambda_2(f) < 9$ *then* F *is fibred by hyperelliptic, trigonal or tetragonal curves.*

(ii) *If* $\lambda_2(f) < 8$ *then* F *is fibred by genus* 3 *or hyperelliptic curves.*

(iii) *If* $\lambda_2(f) < \frac{16}{3}$ *then* F *is fibred by genus* 2 *curves.*

Corollary 3.5. *With the same notations as in Theorem* 3.3*, if* $\mathcal{E} = f_*\omega_{T/B}$ *has a quotient of rank one and degree zero, then the same conclusions as in Theorem* 3.3 *hold.*

In particular, if $b = 0, 1$ *and* F *is not fibred by d-gonal curves (*$d = 2, 3, 4$*) and* $\lambda_2(f) < 9$ *then* $\mathcal{E}$ *is ample.*

Proof. According to Proposition 1.8 (ii), if $\mathcal{E}$ has a quotient $\mathcal{L}$ of rank 1 and degree zero, then it is torsion and so it is trivial after an étale base change $\sigma : \widetilde{B} \longrightarrow B$. Using now part (i) of Proposition 1.18, the induced new fibration $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{B}$ verifies $q(\widetilde{T}) > g(\widetilde{B})$ and hence Theorem 3.3 applies. Finally note that both fibrations have the same slope λ_2 by Lemma 1.5 (ii). □

Remark 3.6. Although in Theorem 2.4 we only must take care of subpencils of $|K_F|$, in the proof of Theorem 3.3 we must take care of subpencils of $|K_{\widetilde{F}}|$ for any étale cover $\widetilde{F} \longrightarrow F$, hence corresponding to *arbitrary* pencils in F. Hence the hypotheses that appear in the statement of Theorem 3.3 can not be restricted to subpencils of $|K_F|$.

Example 3.7. We give a family of examples of fibred threefolds with F fibred by genus two curves and with slope arbitrarily near to 6. For this, consider a ruled surface R onto a smooth curve C of genus m, and let B be a smooth curve of genus b. Let $Y = R \times B$ and consider a suitable double cover $T \longrightarrow Y$. If the ramification locus is suitably chosen, a general fibre F of the induced fibration $f : T \longrightarrow B$ has a genus two fibration. A standard computation shows that $\lambda_2(f)$ is arbitrarily near to 6 provided $m \geq 1$ (in fact equal to 6 if $m = 1$). Observe that by construction $q(T) - b \geq m$ and so

f is a non Albanese fibration. Thus, we can conclude that the bound 6 has certainly some meaning for fibrations with general fibre fibred by hyperelliptic curves. The same construction produces examples with arbitrary $g \geq 3$ but then $\lambda_2(f)$ is far from 6.

4. Fibred threefolds with low slope

In [29], the following possibilities for fibred threefolds with fibre of general type and $\lambda_2(f) < 4$ are listed.

Theorem 4.1. (Ohno, [29] Main Theorem 2). *Let $f : T \longrightarrow B$ be a relatively minimal fibred threefold as in Theorem 2.4. Assume F is of general type. If $K^3_{T/B} < 4(\chi(\mathcal{O}_B)\chi(\mathcal{O}_F) - \chi(\mathcal{O}_T))$ then F has one of the following properties:*

(i) *F carries a linear pencil of curves of genus two.*

(ii) $K^2_F \leq 2p_g(F) - 1$

(iii) *$K^2_F = 2p_g(F)$, $p_g(F) \geq 3$, $q(F) \leq 2$ and $|K_F|$ is not composed ($q(F) = 2$ only if $p_g(F) = 3$).*

(iv) *$|K_F|$ is not composed and*

- $K^2_F = 8$, $p_g(F) = 3$, $q(F) \leq 1$ *or*
- $K^2_F = 9$, $p_g(F) = 4$, $q(F) \leq 1$ *or*
- $K^2_F = 7$, $p_g(F) = 3$, $q(F) \leq 2$

(v) *$K^2_F = 4$ or 5, $p_g(F) = 2$ and the movable part of $|K_F|$ is a linear pencil of curves of genus three with only one base point.*

(vi) *$K^2_F = 2$ or 3 and $p_g(F) = 1$.*

(vii) $p_g(F) = 0$.

Moreover Ohno gives an example of fibration of type (i).

In the case of fibred surfaces, no examples are known with slope less than 4 and non-hyperelliptic general fibre if $g = g(F) >> 0$. It seems rather plausible that if the genus of the fibre is large enough, then the general fibre must be hyperelliptic. The analogy in the case of threefolds is clear: the canonical map $\varphi_{|K_F|}$ should not be *general*. Curiously enough, we can prove this. If $p_g(F) \leq 2$ the canonical map is clearly very special. We can prove that if $p_g(F) \geq 8$, F is fibred by hyperelliptic curves (in fact of genus 2 if $p_g(F) \geq 15$) and hence $\varphi_{|K_F|}$ has at least degree two. In the remaining cases $3 \leq p_g(F) \leq 7$ we also prove that the canonical map of F has degree 3 up to some exceptions.

In fact what happens is that only the first case in Ohno's classification occurs when $p_g(F)$ is large enough. More concretely:

Theorem 4.2. *Let $f : T \longrightarrow B$ be a relatively minimal fibration of a normal, projective threefold T with only canonical singularities onto a smooth curve B of genus b. Assume that the general fibre F is of general type, $p_g(F) \geq 3$ and $\chi_f = \chi(\mathcal{O}_F)\chi(\mathcal{O}_B) - \chi(\mathcal{O}_T) > 0$.*

Then, if $\lambda_2(f) < 4$, we have:

(i) $q(T) = b$

(ii) $\mathcal{E} = f_*\omega_{T/B}$ *has no invertible rank zero quotient sheaf (in particular, $\mathcal{E}$ is ample provided $b \leq 1$).*

(iii) *If $p_g(F) \geq 15$ then F has a rational pencil of curves of genus 2.*

(iv) *If $p_g(F) \leq 14$ then one of the following holds*

  (a) *F has a rational pencil of hyperelliptic curves*

  (b) *F has a rational pencil of trigonal curves, $q(F) = 0$ and*

   . *either the canonical map of F is of degree 3 and either $p_g(F) = 3$, $3 \leq K_F^2 \leq 8$ or $p_g(F) = 4, 5$ $3p_g(F) - 6 \leq K_F^2 \leq 9$*

   . *or the canonical map of F is birational and*

$$p_g(F) = 4 \quad 5 \leq K_F^2 \leq 9$$

$$5 \leq p_g(F) \leq 7 \quad 3p_g(F) - 7 \leq K_F^2 \leq 2p_g(F).$$

  (c) *F is the quintic surfce in $\mathbb{P}^3$ (that is, F is canonical, $p_g(F) = 4$, $q(F) = 0$, $K_F^2 = 5$).*

Remark 4.3. It is doubtful that the cases of fibre F canonical in (iv)(b) and in (iv)(c) occur. In [1] the author proves that then $K_{T/B}^3 \geq 4\chi_f$, provided T is Gorenstein, so any example should necessarily have T non Gorenstein.

Proof.

The first two statements follow from Theorem 3.3 and Corollary 3.5. Following the list of Ohno in Theorem 4.1, if $\lambda_2(f) < 4$ and $p_g(F) \geq 3$, $|K_F|$ is not composed.

We follow the notations of the proof of Theorem 2.4.

Assume F has a rational hyperelliptic pencil. We must prove that the pencil is of genus 2 provided $p_g(F) \geq 15$. Put $\delta = K_F\widehat{D}$. If the (geometric) genus of $\widehat{D}$ is not 2 then we observe that $\delta \geq 4$ (if $\widehat{D}^2 = 0$ then $\delta = 2g - 2$; if $\widehat{D}^2 > 0$ Hodge index theorem gives $\delta^2 \geq K_F^2 \geq 2p_g(F) - 4 \geq 26$). Formula (17) reads for $r = 2, p = 0$

$$\begin{aligned} K_{T/B}^3 &\geq 2\delta\Delta_1 + 4\Delta_2 - \delta(\mu_1 - \mu_{m-1}) - 2\delta(\mu_{m-1} - \mu_m) - 6\mu_m - 2\mu_\ell \\ &\geq 2\delta\Delta_1 + 4\Delta_2 - 2\delta\mu_1 \end{aligned}$$

and using that $K^3_{T/B} \geq (2p_g(F)-4)(\mu_1+2\mu_\ell)$ we get

$$(1+\frac{2\delta}{2p_g(F)-4})K^3_{T/B} \geq 2\delta\Delta_1 + 4\Delta_2$$

which together with (19) gives that $K^3_{T/B} \geq 4\Delta_f$ provided $p_g(F) \geq 15$.

Assume F has no rational hyperelliptic pencil. According to Remark 2.3 we must check when the coefficient of $(\mu_i - \mu_{i+1})$ is greater than or equal to $4r_i$.

Take i such that $m \leq i \leq \ell - 1$. If $a_i = 2$, then $P_i^2 \geq 2r_i - 2$ if the image is ruled (since F has no hyperelliptic rational pencil, $\deg\varphi_i(F) \geq r_i - 1$) or $P_i^2 \geq 4r_i - 8$ otherwise. In any case $P_i^2 \geq 2r_i - 2$ ($r_i \geq 3$ since $|P_i|$ is not composed), and hence $P_i^2 + P_iP_{i+1} + P_{i+1}^2 \geq 3P_i^2 \geq 6r_i - 6 \geq 4r_i$.

If $a_i = 3$, then $a_{i+1} = 1$ or 3 and hence $P_{i+1}^2 \geq 3r_{i+1} - 7 \geq 3r_i - 4$. If $P_i \neq P_{i+1}$, then $P_i^2 + P_iP_{i+1} + P_{i+1}^2 \geq 2P_i^2 + 1 + P_{i+1}^2 \geq 9r_i - 15 \geq 4r_i$. If $P_i = P_{i+1}$, then $P_i^2 + P_iP_{i+1} + P_{i+1}^2 = 3P_{i+1}^2 \geq 9r_i - 12 \geq 4r_i$.

If $a_i \geq 4$, $P_i^2 + P_iP_{i+1} + P_{i+1}^2 \geq 3P_i^2 \geq 12r_i - 24 \geq 4r_i$. Finally if $a_i = 1$, then $a_{i+1} = 1$ and hence, by the same argument as in (9), $P_i^2 + P_iP_{i+1} + P_{i+1}^2 \geq 9r_i - 17 \geq 4r_i$ (since $r_i \geq 4$ if φ_i is birational).

Let $i = \ell$. As pointed out in Remark 2.3, we can set $P_{\ell+1} = \tau^*K_F$. Hence, if $P_\ell = P_{\ell+1} = \tau^*K_F$, we have $P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 = 3P_\ell^2$ but if $P_{\ell+1} \neq P_\ell$, we have $P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 \geq K_F^2 + 2P_\ell^2 + 1 \geq 3P_\ell^2 + 2$. Keeping this in mind we obtain that $P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 \geq 4r_\ell$ except when $r_\ell = p_g(F) = 4$, $P_\ell^2 = K_F^2 = 3p_g - 7 = 5$ and F is canonical or $r_\ell = p_g(F) = 3$, $P_\ell^2 = 3p_g - 6 = 3$, $K_F^2 = 3p_g(F) - 6$ or $3p_g(F) - 5$ and the canonical map is of degree three. In both cases we necessarily have $q(F) = 0$ (see [1] for the canonical case and [37] for the degree 3 case).

Take i such that $1 \leq i \leq m - 1$. If $r_i = 1$ (then $i = 1$ and $P_1 = 0$), we have $(P_1 + P_2)P_m \geq 4r_1 = 4$ except when $r_2 = 2$ and P_m induces a g_3^1 in the fibre of the rational pencil $|P_2|$. Assume $r_i \geq 2$. If $a_m = 2$, then $P_iP_m \geq 2r_i$; for this we must look at the proof of Lemma 2.1 (ii). Assume $2r_i - 1 \geq P_iP_m \geq \alpha_2(\alpha_1 ad) \geq (\alpha_1 ad)(r_i - 1)$; we have that $|P_m|_{|D} = g_2^1$, hence $a = 2$ and necessarily $\alpha_1 = d = 1$; if $\alpha_2 = r_i - 1$ the pencil would be rational (since $\alpha_1 = 1$) which is impossible by our assumptions; hence $\alpha_2 \geq r_i$ which is again impossible. Then $(P_i + P_{i+1})P_m \geq 2P_iP_m \geq 4r_i$. If $a_m \geq 3$, then by Lemma 2.1 $P_iP_m \geq 3(r_i - 1) \geq 2r_i$ except if $r_i = 2$, $a_m = 3$. In this exceptional case, if $P_{i+1} \neq P_m$, then $(P_i + P_{i+1})P_m \geq (3r_i - 3) + (3r_{i+1} - 3) \geq 6r_i - 3 = 9 > 8 = 4r_i$; if $P_{i+1} = P_m$, then $(P_i + P_m)P_m \geq 8 = 4r_i$ except if $4 \geq P_m^2 \geq 3r_m - 6$, i.e., $r_1 = 2$, $r_2 = 3$, $m = 2$, $a_m = 3$ (which again produces a rational trigonal pencil in F). Finally if $a_m = 1$, then $P_iP_m \geq 4r_i - 4$ (Lemma 2.1) and hence $P_iP_m \geq 2r_i$ and $(P_i + P_{i+1})P_m \geq 4r_i$.

So we can conclude than either $p_g(F) = 4$, $q(F) = 0$, $K_F^2 = 5$ or F has a rational trigonal pencil.

Note that in the discussion above, when F has a rational trigonal pencil, $|P_m|$ induces a degree 3 map. Hence the canonical map of F can only be of degree 1 or 3. In

any case $K_F^2 \geq 3p_g(F) - 7$. Hence, applying Theorem 4.1 we have $3p_g(F) - 7 \leq K_F^2 \leq 2p_g(F)$ (if $p_g(F) \geq 5$) and hence $p_g(F) \leq 7$, $K_F^2 \leq 2p_g(F) \leq 14$.

Finally we must prove that $q(F) = 0$. If $q(F) = 1$, then $K_F^2 \geq 3p_g(F) + 7q(F) - 7 = 3p_g(F)$ (cf. [21]) which is impossible. Assume $q(F) \geq 2$. If $|K_F|$ is birational, we have $3p_g(F) - 4 \leq K_F^2 \leq 2p_g(F) - 1$ (if $q(F) \geq 1$, we have $K_F^2 \geq 3p_g(F) + q(F) - 7$ but if equality holds then $q(F) \geq 3$ (cf., e.g., [1])) which is impossible. If $|K_F|$ induces a map of degree 3, we have $3p_g(F) - 3 \leq K_F^2 \leq 2p_g(F)$ (cf. [37] and Theorem 4.1) or $p_g(F) = 3$, $q(F) = 2$, $K_F^2 = 7$; so in any case we get $p_g(F) = 3$, $K_F^2 \geq 6$. Following the above discussion the only possibilities for the Harder-Narasimhan filtration of $\mathcal{E}$ are $r_1 = 2$, $r_2 = 3$ or $r_1 = 1$, $r_2 = 2$, $r_3 = 3$. The first one gives

$$K_{T/B}^3 \geq (P_1 + P_2)P_2(\mu_1 - \mu_2) + 3P_2^2\mu_2 \geq 9(\mu_1 - \mu_2) + 18\mu_2 \geq 8(\mu_1 - \mu_2) + 12\mu_2 = 4\Delta_f.$$

The last one gives

$$\begin{aligned} K_{T/B}^3 &\geq (P_1 + P_2)P_3(\mu_1 - \mu_2) + (P_2 + P_3)P_3(\mu_2 - \mu_3) + 3P_3^2\mu_3 \\ &\geq 3(\mu_1 - \mu_2) + 9(\mu_2 - \mu_3) + 18\mu_3 \geq 4(\mu_1 - \mu_2) + 8(\mu_2 - \mu_3) + 12\mu_3 = 4\Delta_f \end{aligned}$$

if $\mu_2 - \mu_3 \geq \mu_1 - \mu_2$; otherwise consider

$$\begin{aligned} K_{T/B}^3 &\geq (P_1 + P_3)P_3(\mu_1 - \mu_3) + 3P_3^2\mu_3 \\ &\geq 6(\mu_1 - \mu_3) + 18\mu_3 \geq 4(\mu_1 - \mu_2) + 8(\mu_2 - \mu_3) + 12\mu_3 = 4\Delta_f. \end{aligned}$$

So we necessarily have $q(F) = 0$.

As for the restrictions for $(p_g(F), K_F^2)$ when the canonical map is of degree 3, we refer to [30], [37], [20]. □

References

[1] M.A.Barja, *Numerical bounds of canonical varieties*, Preprint (1997).

[2] M.A.Barja, *On the slope and geography of fibred surfaces and threefolds*, Ph. D. Thesis (1998).

[3] M. A. Barja; J.C. Naranjo, *Extension of maps defined on many fibres*, To appear in Collectanea Math.

[4] A. Beauville, *Surfaces algébriques complexes*, Société Mathématique de France. Astérisque **54** (1978).

[5] A. Beauville, *L'application canonique pour les surfaces de type général*, Invent. Math. **55** (1979), 121-140.

[6] A. Beauville, *Annullation du H^1 et systèmes paracanoniques sur les surfaces*, J. Reine Math. **388** (1988), 149-157.

[7] J.-X. Cai, *Irregularity of certain algebraic fiber spaces*, Manuscripta Math. **99** (1998), 273-287.

[8] R. E. Elkik, *Rationalité des singularités canoniques*, Invent. Math. **64** (1981), 1-6.

[9] E. Freitag, *Uber die struktur der funktionenkorper zu hyperabelschen gruppen I*, J. Reine Angew. Math. **247** (1971), 97-117.

[10] T. Fujita, *On Kahler fiber spaces over curves*, J. Math. Soc. Japan **30** (1978), 779-794.

[11] T. Fujita, *The sheaf of relative canonical forms of a Kahler fiber space over a curve*, Proc. Japan Acad. **54** (1978), 183-184.

[12] W. Fulton, *Intersection Theory*, Ergeb. Math. Grenzg. **2**

[13] M. Green, R. Lazarsfeld, *Deformation theory, generic vanishing theorems and some conjectures of Enriques, Catanese and Beauville*, Inv. Math. **90** (1987), 389-407.

[14] B. Hunt, *Complex manifold geography in dimension 2 and 3*, J. Differential Geom. **30** (1989), 51-153.

[15] Y. Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degeneration of surfaces*, Ann. of Math. **127** (1988), 93-163.

[16] G.Kempf, F. Knudsen, D. Mumford, B.Saint-Donat, *Toroidal embeddings I*, LNM **339** (1973) Springer.

[17] J. Kollár, *Subadditivity of the Kodaira dimension: fibers of general type*, Adv. studies in Pure Math. **10** (1987), Algebraic Geometry, Sendai 1985. 361-398.

[18] J. Kollár, *Higher direct images of sheaves I*, Ann. of Math. **123** (1986), 11-42.

[19] J. Kollár, *Higher direct images of dualizing sheaves II*, Ann. of Math. **124** (1986), 171-202.

[20] K. Konno. *Algebraic surfaces of general type with* $c_1^2 = 3p_g - 6$, Math. Ann. **290** (1991), 77-107.

[21] K. Konno, *A note on surfaces with pencils of non-hyperelliptic curves of genus* 3, Osaka J. Math. **28** (1991), 737-745.

[22] K. Konno, *On the irregularity of special non-canonical surfaces*, Publ. RIMS Kyoto Univ. **30** (1994), 671-688.

[23] K. Konno, *A lower bound of the slope of trigonal fibrations*, Internat. J. Math. vol. **7** no. 1 (1996), 19-27.

[24] K. Konno, *Clifford index and the slope of fibred surfaces*, Preprint 1997.

[25] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic Geometry, Sendai (1985); Adv. Study in Pure Math. **10** (1987), 449-476.

[26] S. Mori, *Classification of higher-dimensional varieties*, Proc. Symp. in Pure Math. **46** (1987).

[27] S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. **1** (1988), 117-253.

[28] M. Mendes Lopes, *The degree of the generators of the canonical ring of surfaces of general type with* $p_g = 0$, Arch. Math. **69** (1997), 435-440.

[29] K. Ohno, *Some inequalities for minimal fibrations of surfaces of general type over curves*, J. Math. Soc. Japan **44** (1992), 643-666.

[30] G. Pompilj, *Sulle superficie algebriche le cui curve canoniche posseggono una* g_3^1, Rend. Reale Ist. Lombardo di Sc. e Lett. **74** (1941), 280-286.

[31] F. Serrano, *Isotrivial fibred surfaces*, Annali Mat. Pura ed appl. **IV** (1996), 63-81.

[32] C. Simpson, *Subspaces of moduli spaces of rank one local systems*, Ann. scient. Éc. Norm. Sup. 4è série, t.26 (1993), 361-401.

[33] Z.E. Stankova-Frenkel, *Moduli of trigonal curves*, Preprint 1997.

[34] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Berlin, Heidelberg, New York, Springer (1995).

[35] G. Xiao, *L'irregularité des surfaces de type général dont le système canonique est composé d'un pinceau*, Comp. Math. **56** (1985), 251-257.

[36] G. Xiao, *Fibred algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449-466.

[37] F. Zucconi, *Surfaces with canonical map of degree three and* $K^2 = 3p_g - 5$. Preprint.